Stokes solution for initializing swirling vortex ring in bounded flow

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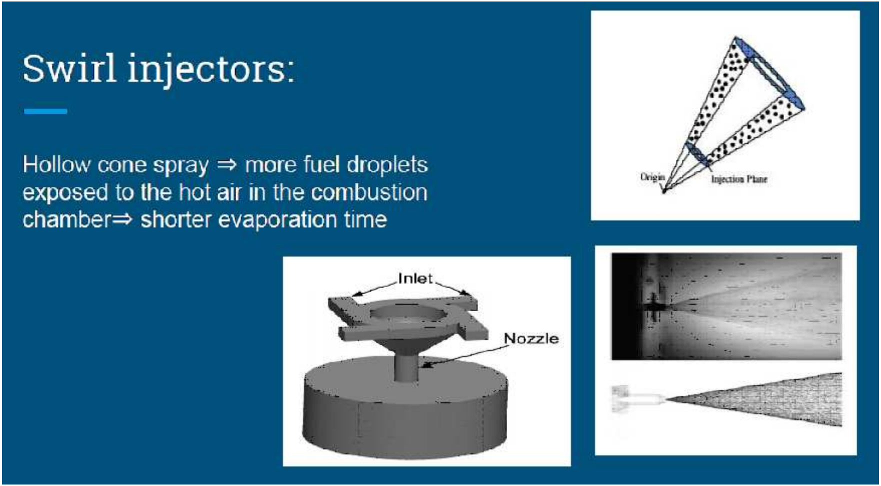
Motivation

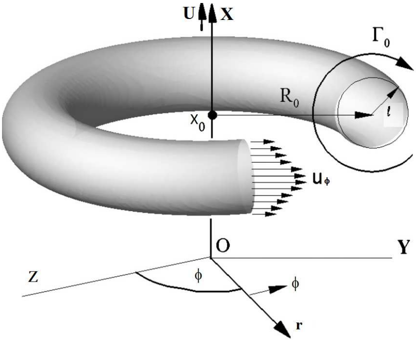
"Vortex rings with an azimuthal velocity component are an essential building block of turbulent flows."(Gargan - Shingles, Rudman and Ryan, 2015).



Outline

- Previous methods for initializing SVVR (swirling viscous vortex ring) in numerical simulations
- Formulation of the problem for SVVR in bounded flow
- Stokes solutions for SVVR in unbounded and bounded flow
- Idea to initialize SVVR in unbounded and bounded flow based on the obtained solutions
- Analytical results versus numerical data for unbounded SVVR

Scheme of a swirling vortex ring



Scheme of a swirling vortex ring in a tube

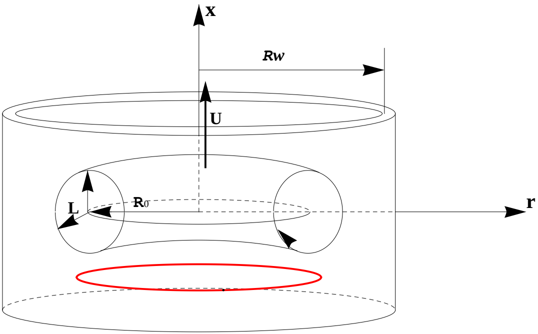


Figure: Schematic of a swirling vortex ring in a tube.

Previous work on vortex rings

- Early work: Helmholtz (1858), Kelvin (1869), Hill (1894), Lamb(1916)

- Evolution and formation: Fraenkel (1972), Norbury (1973), Saffman (1975, 1978), Maxworthy(1972, 1974, 1977), Moore (1980), Kaltaev (1983), Rott&Cantwell (1993), Gharib et. al (1998), Fukumoto&Moffatt(2000,2008)

Swirling vortex rings

- Usually the initializing of swirling vortex ring is provided by employing of the Gaussians distributions for vorticity and swirling velocity (Virk *et al.*, 1992, Cheng *et al.*, 2010, Cargan *et al.*, 2015).
- This methodology is based on the fact that this distribution for the vorticity was used for deriving of the Saffman's velocity formula (Saffman 1970, 1992) for thin rings and its validity for starting phase of ring's development was proved many times both numerically (Sullivan *et al.* 1973) and experimentally (Weigand *et al.*, 1997, Donnelly *et al.*, 1991)
- Alternate way: Firstly, to model the piston/cylinder vortex generator by simulating the flow in the cylinder or to prescribe the axial velocity profile at the inflow boundary (Danaila and Helio, 2008). Secondary, when the pinch-off of the vortex ring will be over to include into consideration the swirl. The latter method to my knowledge was not reported in literature.

Swirling vortex rings: goal of the present work

- The idea behind our method (Fukumoto and Kaplanski, 2010, 2013) is to use for this purpose the solution to the Navier - Stokes equation for the axisymmetric geometry and arbitrary times, developed in the papers (Kaplanski *et al.*,1999,2005). As shown in (Fukumoto and Kaplanski, 2008) obtained on the basis of this solution the expressions for the kinetic energy and translational velocity of the vortex ring are identical with the Saffman's formulae (particularly it is easy to show with the help of the Mathematica).

Swirling vortex rings: benefits of the present formulation

- The considered solution overcomes the inconsistency with the Gaussian vorticity, which do not satisfy the continuity equation, and expand the prediction of the real vorticity field for a greater time interval. Moreover, this solution allows to obtain in the closed form the integral characteristics of the considered flow like the circulation, kinetic energy, translational velocity, enstrophy and helicity.
- We also consider the generalization of this solution for the swirling vortex ring moving inside the tube (Danaila, Kaplanski and Sazhin, 2015). In our model the streamfunction corresponding to abovementioned solution for the vorticity is obtained in the modified form, which allows to satisfy the condition of no flow on the wall of the tube.
- Finally, it is shown that in the present formulation, the time - dependent integral characteristics of the swirling viscous vortex ring in a tube can be obtained taking into account the viscosity, swirl and confinement effects.

Full Navier-Stokes equations in cylindrical coordinates

$$q_\theta = v_\theta, \quad q_r = r v_r, \quad q_z = v_z$$

$$\frac{\partial q_r}{\partial r} + \frac{\partial q_\theta}{\partial \theta} + r\frac{\partial q_z}{\partial z} = 0.$$

$$\left.\begin{aligned}
\frac{Dq_\theta}{Dt} &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}\left[\left(\frac{1}{r}\frac{\partial r q_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 q_\theta}{\partial \theta^2} + \frac{\partial^2 q_\theta}{\partial z^2} + \frac{2}{r^3}\frac{\partial q_r}{\partial \theta}\right], \\
\frac{Dq_r}{Dt} &= -r\frac{\partial p}{\partial r} + \frac{1}{Re}\left[r\left(\frac{1}{r}\frac{\partial q_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 q_r}{\partial \theta^2} + \frac{\partial^2 q_r}{\partial z^2} - \frac{2}{r}\frac{\partial q_\theta}{\partial \theta}\right], \\
\frac{Dq_z}{Dt} &= -\frac{\partial p}{\partial z} + \frac{1}{Re}\left[\frac{1}{r}\left(r\frac{\partial q_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 q_z}{\partial \theta^2} + \frac{\partial^2 q_z}{\partial z^2}\right],
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{Dq_\theta}{Dt} &\equiv \frac{\partial q_\theta}{\partial t} + \frac{1}{r}\frac{\partial q_\theta q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta^2}{\partial \theta} + \frac{\partial q_\theta q_z}{\partial z} + \frac{q_\theta q_r}{r^2}, \\
\frac{Dq_r}{Dt} &\equiv \frac{\partial q_r}{\partial t} + \frac{\partial}{\partial r}\left(\frac{q_r^2}{r}\right) + \frac{\partial}{\partial \theta}\left(\frac{q_\theta q_r}{r}\right) + \frac{\partial q_r q_z}{\partial z} - q_\theta^2, \\
\frac{Dq_z}{Dt} &\equiv \frac{\partial q_z}{\partial t} + \frac{1}{r}\frac{\partial q_r q_z}{\partial r} + \frac{1}{r}\frac{\partial q_\theta q_z}{\partial \theta} + \frac{\partial q_z^2}{\partial z}.
\end{aligned}\right\}$$

Taken from Verzicco and Orlandi (1996), for our case should be replaced $x \to z$

Swirling vortex ring in a tube: Governing equations assuming axisymmetrical flow

The governing equations for the vorticity (ω), streamfunction (Ψ) and swirl velocity (u_ϕ) can be presented as:

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial r}\left(-\frac{1}{r}\frac{\partial \Psi}{\partial x}\omega\right) + \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\omega\right) = \nu[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \omega r}{\partial r}) + \frac{\partial^2 \omega}{\partial x^2}] + \frac{1}{r}\frac{\partial {u_\phi}^2}{\partial x}, \tag{1}$$

$$\frac{\partial u_\phi}{\partial t} + \frac{1}{r}\frac{\partial \Psi}{\partial r}\frac{\partial u_\phi}{\partial x} - \frac{1}{r}\frac{\partial \Psi}{\partial x}\frac{\partial u_\phi}{\partial r} - \frac{1}{r}\frac{\partial \Psi}{\partial x}\frac{u_\phi}{r} = \nu[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial u_\phi r}{\partial r}) + \frac{\partial^2 u_\phi}{\partial x^2}], \tag{2}$$

where x, r are the axes of a cylindrical coordinate system and t is time. The streamfunction is related to the vorticity by

$$\omega = \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \Psi}{\partial r}) + \frac{1}{r}\frac{\partial^2 \Psi}{\partial x^2}, \tag{3}$$

and the velocities are related to the streamfunction by

$$u = -\frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad v = \frac{1}{r}\frac{\partial \Psi}{\partial x}, \tag{4}$$

Swirling vortex ring in a tube: Boundary conditions

We consider the following boundary conditions:
symmetry at the axis:

$$\omega(0,x) = u_\phi(0,x) = \Psi(0,x) = 0, \quad \text{for} \quad r = 0, \tag{5}$$

vanishing vorticity, swirl velocity and stream function in the far field of the vortex:

$$\omega, u_\phi, \Psi \to 0 \quad \text{when} \quad \left(x^2 + r^2\right)^{1/2} \to \infty, \quad \text{for} \quad r < R_w, \tag{6}$$

and no flow through the tube wall:

$$\omega, u_\phi \to 0, \quad \frac{1}{r}\frac{\partial \Psi}{\partial x} = 0, \quad \text{for} \quad r = R_w. \tag{7}$$

Swirling vortex ring in a tube: Governing equations in the other form

We can also rewrite the governing equations in the following form:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial r} - \frac{v\omega}{r} = \nu[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \omega r}{\partial r}) + \frac{\partial^2 \omega}{\partial x^2}] + \frac{1}{r}\frac{\partial u_\phi^2}{\partial x}, \quad (8)$$

$$\frac{\partial u_\phi}{\partial t} + u\frac{\partial u_\phi}{\partial x} + v\frac{\partial u_\phi}{\partial r} + \frac{u u_\phi}{r} = \nu[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial u_\phi}{\partial r}) + \frac{\partial^2 u_\phi}{\partial x^2}], \quad (9)$$

Swirling vortex ring in a tube: Restrictions and invariants

- Note that governing equations with considered boundary conditions can describe swirling vortex rings in a tube as long as the viscous interaction of the rings with the tube wall remains weak and can be ignored. For longer time instants, when this assumption is not valid, the boundary condition of no flow on the wall should be replaced with a non-slip wall condition.
- Also for high Reynolds numbers and significant viscous interaction of the rings with a wall, we should to employ the system of the full Navier-Stokes equations (or, by other words, to refuse from an assumption about axisymmetric flow) with the purpose to take into account the flow friction against the tube wall.

The momentums of the vorticity and swirl velocity defined as

$$M = \pi \int_{-\infty}^{\infty} \int_{0}^{R_w} r^2 dr dx, \quad J = 2\pi \int_{-\infty}^{\infty} \int_{0}^{R_w} u_\phi r^2 dr dx \tag{10}$$

should conserves with time.

Swirling vortex ring in a tube: Dimensionless form of the governing equations

$$x_1 = \frac{x}{R_0}, \quad r_1 = \frac{r}{R_0}, \quad t_1 = \frac{t\nu}{R_0^2}, \quad \zeta = \frac{\omega}{\omega_0}, \quad \Psi^*_{VRC} = \frac{\Psi}{\omega_0 R_0^3}, \ W = \frac{u_\phi}{u_\phi^0}. \tag{11}$$

$$\frac{\partial \zeta}{\partial t_1} + Re[-\frac{\partial}{\partial r_1}[\frac{\zeta}{r_1}\frac{\partial}{\partial x_1}] + \frac{\partial}{\partial x_1}[\frac{\zeta}{r_1}\frac{\partial}{\partial r_1}]]\Psi^*_{VRC} = [\frac{\partial}{\partial r_1}(\frac{1}{r_1}\frac{\partial \zeta r_1}{\partial r_1}) + \frac{\partial^2 \zeta}{\partial x_1^2}]$$

$$+ReS(\frac{1}{r_1}\frac{\partial W^2}{\partial x_1}), \tag{12}$$

$$\zeta = \frac{\partial}{\partial r_1}(\frac{1}{r_1}\frac{\partial \Psi^*_{VRC}}{\partial r_1}) + \frac{1}{r_1}\frac{\partial^2 \Psi^*_{VRC}}{\partial x_1^2}, \tag{13}$$

$$\frac{\partial W}{\partial t_1} + Re[\frac{1}{r_1}\frac{\partial \Psi^*_{VRC}}{\partial r_1}\frac{\partial W}{\partial x_1} - \frac{1}{r_1}\frac{\partial \Psi^*_{VRC}}{\partial x_1}\frac{\partial W}{\partial r_1} - \frac{1}{r_1}\frac{\partial \Psi^*_{VRC}}{\partial x_1}\frac{W}{r_1}]$$

$$= [\frac{\partial}{\partial r_1}(\frac{1}{r_1}\frac{\partial W r_1}{\partial r_1}) + \frac{\partial^2 W}{\partial x_1^2}], \tag{14}$$

Swirling vortex ring in a tube: Dimensionless parameters

- Reynolds number is $Re = \omega_0 R_0^2/\nu$; parameter that measure swirl (Swirling number is the ratio of the axial flux of angular momentum to the axial flux of the axial momentum) $S = (u_\phi{}^0)^2/\omega_0{}^2 R_0^2$ and the parameter $\varepsilon = R_0/R_w < 1$ that quantifies the confinement of the vortex ring. The wall is located at ($r_{1w} = 1/\varepsilon$).
- We restrict our attention to a vortex ring in the low-Reynolds-number motion and, in what follows, ignore the nonlinear terms and also assume that S is small ($S << 1$).

Swirling vortex ring in a tube: Solution

The appropriate linearized equations can be solved subject to the initial condition

$$\zeta_0 = \delta(x_1)\delta(r_1 - 1), \quad W_0 = \delta(x_1)\delta(r_1 - 1), \tag{15}$$

where δ is the Dirac delta function. Solution of the time-dependent Stokes equation:

$$\zeta = \exp(-(k^2 + m^2)t_1)\exp(ikx_1)\mathrm{J}_1(mr_1), \tag{16}$$

where k and m are arbitrary constants. Taking into account that ζ vanishes as $r \to \infty$ and using a synthesis of all of the terms with $\infty > m \geq 0$ and $-\infty < k < \infty$, we can get a general solution

$$\zeta(x_1, r_1, t_1) = \int_{-\infty}^{\infty}\int_0^{\infty} a(k, m)\exp(-(k^2+m^2)t_1)\exp(ikx_1)\mathrm{J}_1(mr_1)dmdk. \tag{17}$$

Using the Fourier–Bessel theorem and $\zeta(x', r', 0) = \zeta_0(x', r')$,

$$a(k, m) = \frac{m}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \zeta_0(x', r')\exp(-ikx')\mathrm{J}_1(mr')r'dr'dx'. \tag{18}$$

Using the initial condition, we can obtain $a(k, m)$ in the following form

$$a(k,m) = \frac{m}{2\pi} \mathrm{J}_1(m). \tag{19}$$

By employing (Prudnikov *et al.*, 1983, the integral 2.12.39(3))

$$\int_0^\infty m \exp(-(m^2)t_1) \mathrm{J}_1(m) \mathrm{J}_1(mr_1) dm = \frac{1}{2t_1} \exp(-\frac{r_1^2+1}{4t_1}) \mathrm{I}_1(\frac{r_1}{2t_1}), \tag{20}$$

and

$$\int_{-\infty}^\infty \exp(ikx_1) \exp(-(k^2)t_1) dk = \frac{1}{2\pi} \exp(-\frac{(x_1)^2}{4t_1}) \frac{\sqrt{\pi}}{\sqrt{t_1}}. \tag{21}$$

$$\zeta(x_1, r_1, \theta) = \frac{\theta^3}{\sqrt{2\pi}} \exp(-\frac{(x_1^2 + r_1^2 + 1)\theta^2}{2}) \mathrm{I}_1(r_1\theta^2),$$

$$W(x_1, r_1, \theta) = \frac{\theta^3}{\sqrt{2\pi}} \exp(-\frac{(x_1^2 + r_1^2 + 1)\theta^2}{2}) \mathrm{I}_1(r_1\theta^2), \tag{22}$$

where $\theta = 1/\sqrt{2t_1} = R_0/\sqrt{2\nu t} = R_0/L$ designates dimensionless time and I_1 is modified Bessel function of order one.

Gaussian distribution of vorticity

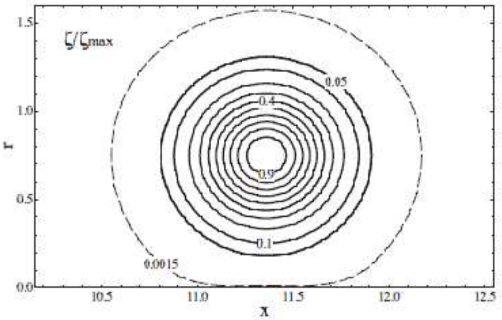


FIG. 1. Isocontours of the normalized vorticity ζ/ζ_{max} for the values $\theta = 3.56$, $R_0 = 0.783$, and $X_0 = 11.36$, that give best fit of the vortex (Eq. (1)) to the simulated vortex (Ref. 13). The dashed line represents contour for $\zeta/\zeta_{max} = 0.0015$.

Following (Danala, Kaplanski and Sazhin, 2015) the streamfunction corresponding to a confined vortex ring can be expressed as

$$\Psi^*_{VRC} = \Psi^*_{VR} - \Psi_0 = \{\frac{r_1\theta}{4}\int_0^\infty F(r_1, x_1, \theta)\mathrm{J}_1(r_1\theta\mu)\mathrm{J}_1(\theta\mu)d\mu\}$$

$$-\{\frac{r_1}{\pi}\int_0^\infty \frac{\mathrm{K}_1(\mu/\varepsilon)}{\mathrm{I}_1(\mu/\varepsilon)}\mathrm{I}_1(r_1\mu)\mathrm{I}_1(\mu)\cos(\mu x_1)d\mu\}. \quad (23)$$

where

$$F(x_1, \mu, \theta) = \exp(x_1\theta\mu)\mathrm{erfc}(\frac{\mu + x_1\theta}{\sqrt{2}}) + \exp(-x_1\theta\mu)\mathrm{erfc}(\frac{\mu - x_1\theta}{\sqrt{2}}),$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt,$$

K_1 is the modified Bessel function of the second kind, Ψ^*_{VR} is the streamfunction in an unbounded flow and Ψ_0 is the streamfunction induced by the presence of the tube.

Swirling vortex rings in a tube: Solution for u, v and u_ϕ

Thus, in cylindrical coordinates (r_1, ϕ, x_1), the initial velocity field for axisymmetric flow can be written in dimensionless form as

$$u = \int_0^\infty \mu[-\frac{\theta^2}{4} F(x_1, \mu, \theta) \mathrm{J}_0(r_1\theta\mu)\mathrm{J}_1(\theta\mu) + \frac{1}{2\pi}\frac{\mathrm{K}_1(\mu/\varepsilon)}{\mathrm{I}_1(\mu/\varepsilon)} \mathrm{I}_0(r_1\mu)\mathrm{I}_1(\mu)\cos(\mu x_1)]d\mu, \quad (24)$$

$$v = \int_0^\infty \mu[\frac{\theta^2}{4}\tilde{F}(x_1, \mu, \theta)\mathrm{J}_1(r_1\theta\mu)\mathrm{J}_1(\theta\mu) + \frac{1}{2\pi}\frac{\mathrm{K}_1(\mu/\varepsilon)}{\mathrm{I}_1(\mu/\varepsilon)} \mathrm{I}_1(r_1\mu)\mathrm{I}_1(\mu)\sin(\mu x_1)]d\mu, \quad (25)$$

$$u_\phi = \frac{\theta^3}{\sqrt{2\pi}} \exp(-\frac{(x_1^2 + r_1^2 + 1)\theta^2}{2})\mathrm{I}_1(r_1\theta^2), \quad (26)$$

where

$$\tilde{F}(x_1, \mu, \theta) = \exp(x_1\theta\mu)\mathrm{erfc}(\frac{\mu + x_1\theta}{\sqrt{2}}) - \exp(-x_1\theta\mu)\mathrm{erfc}(\frac{\mu - x_1\theta}{\sqrt{2}}).$$

Swirling vortex rings in a tube: Features of the solution

- It is noteworthy to mention that the present model for small time t (when the viscous interaction of the rings with the tube wall is weak) predicts with great accuracy Gaussian profiles for the vorticity and swirl velocity and coincide with the initial profiles employed earlier for the initializing swirling vortex ring at the numerical simulations (Cheng, Lou and Lim, 2010) and (Gargan - Shingles, Rudman and Ryan, 2015).
- In the same time, our profiles for the velocity components (u), (v) and (u_ϕ), contrary to profiles employed in these papers, satisfy the continuity equation.

The dimensional circulation Γ and energy E of the swirling vortex ring in a bounded flow

$$\Gamma = \int_{-\infty}^{\infty} \int_{0}^{R_w} \omega dx dr, \quad E = \pi \int_{-\infty}^{\infty} \int_{0}^{R_w} (\omega \Psi_{VRC} + r {u_\phi}^2) dx dr. \qquad (27)$$

Linear solution predicts immovable ring. However we can estimate its velocity as

$$U = \int_{-\infty}^{\infty} \int_{0}^{R_w} \{ \left(\Psi_{VRC} - 6x \frac{\partial \Psi_{VRC}}{\partial x} \right) \omega - 2 r {u_\phi}^2 \} dr dx \Big/ [2 \int_{-\infty}^{\infty} \int_{0}^{R_w} r^2 \omega dr d \qquad (28)$$

Helmholtz (1858) (see also Lamb,1932; Saffman,1970; 1992), gave meaning to this formula as the velocity of the three-dimensional vortex centroid. This equation is the generalization of the so-called Helmholtz - Lamb transformation formula taking into account the swirl velocity (Saffman, 1970; 1992). Fukumoto and Kaplanski (2010; 2013) suggested to obtain the translational velocity of the swirling viscous vortex ring based on Kaplanski and Rudi solution (1999; 2005).

Swirling vortex rings in a tube: Contribution of confinement and swirl into translational velocity

$$U = U_1 + U_2 + U_3 \tag{29}$$

$$U_1 = \int_{-\infty}^{\infty}\int_0^{R_w}\left\{\left(\Psi_{VR} - 6x\frac{\partial \Psi_{VR}}{\partial x}\right)\omega\right\}drdx \Big/ \left[2\int_{-\infty}^{\infty}\int_0^{R_w} r^2\omega drdx\right]. \tag{30}$$

$$U_2 = \int_{-\infty}^{\infty}\int_0^{R_w}\left\{\left(\Psi_0 - 6x\frac{\partial \Psi_0}{\partial x}\right)\omega\right\}drdx \Big/ \left[2\int_{-\infty}^{\infty}\int_0^{R_w} r^2\omega drdx\right]. \tag{31}$$

$$U_3 = \int_{-\infty}^{\infty}\int_0^{R_w}\left\{-2ru_\phi{}^2\right\}drdx \Big/ \left[2\int_{-\infty}^{\infty}\int_0^{R_w} r^2\omega drdx\right]. \tag{32}$$

Swirling vortex rings in a tube: The part of the translational velocity due to the inertia forces U_1

Using Fourier-Hankel transforms for $r_w \to \infty$:

$$U_1 = \frac{\Gamma_0 \theta \sqrt{\pi}}{4\pi R_0}[3 \exp(-\frac{\theta^2}{2}) I_1 \left(\frac{\theta^2}{2}\right)$$

$$+\frac{\theta^2}{12} {}_2F_2 \left(\{\frac{3}{2}, \frac{3}{2}\}, \{\frac{5}{2}, 3\}, -\theta^2\right) - \frac{3\theta^2}{5} {}_2F_2 \left(\{\frac{3}{2}, \frac{5}{2}\}, \{2, \frac{7}{2}\}, -\theta^2\right)],$$

Saffman's formula at limit for $t \to 0$:

```
In[1]:= U = tau * (3 * Sqrt[Pi] * Exp[-tau * tau / 2] * BesselI[1, tau * tau / 2]
            +
         (1 / 12) * Sqrt[Pi] * tau * tau * HypergeometricPFQ[{3 / 2, 3 / 2}, {5 / 2, 3}, -tau * tau]
        - (3 * Sqrt[Pi] / 5) * tau * tau * HypergeometricPFQ[{3 / 2, 5 / 2}, {2, 7 / 2}, -tau * tau]);
     Series[U, {tau, Infinity, 3}]
```

Out[2]= $e^{-\text{tau}^2}\left(\left(-6\,i + O\left[\frac{1}{\text{tau}}\right]^1\right) + e^{\text{tau}^2}\left(\frac{1}{2}\left(3 - \text{EulerGamma} - 2\,\text{Log}\left[\frac{1}{\text{tau}}\right] - 2\,\text{PolyGamma}\left[0, \frac{3}{2}\right]\right) + O\left[\frac{1}{\text{tau}}\right]^1\right)\right)$

Swirling vortex rings in a tube: The part of the translational velocity due to the inertia forces and confinement $U_1 + U_2$

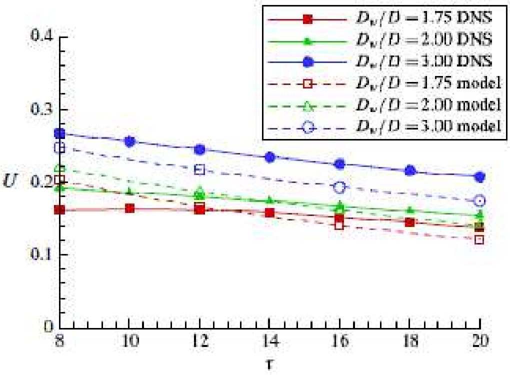


FIGURE 17. (Colour online) Time evolution of the translational velocity U of the vortex ring. Comparison between DNS data and the model predictions (4.5) for three confinement parameters D_w/D. $Re_D = 1700$.

Taken from Danaila, Kaplanski and Sazhin (2015)

Swirling vortex rings in a tube: Contribution of swirl into translational velocity (U_3)

In[1]:=
```
Clear[fp, f, r, x, sum3, R0, L];
f = BesselI[1, (r / L) * R0 / L] * Exp[- (x * x + r * r + R0 * R0) / (2 * L * L)];
sum3 = Integrate[f * f * r, {x, - Infinity, Infinity}, {r, 0, Infinity}]
```

Out[3]=

$$\frac{1}{2} L^2 \,\mathrm{BesselI}\left[1, \frac{R0^2}{2\,L^2}\right]$$

$$\mathrm{If}\left[\mathrm{Re}\left[L^2\right] > 0, \frac{e^{-\frac{R0^2}{2 L^2}}\sqrt{\pi}}{\sqrt{\frac{1}{L^2}}}, \mathrm{Integrate}\left[e^{-\frac{R0^2+2x^2}{2L^2}}, \{x, -\infty, \infty\}, \mathrm{Assumptions} \to \mathrm{Re}\left[L^2\right] \le 0\right]\right]$$

In[4]:=

$$\mathrm{Result} = \mathrm{FullSimplify}\left[\frac{1}{2} L^2 \,\mathrm{BesselI}\left[1, \frac{R0^2}{2\,L^2}\right] * \frac{e^{-\frac{R0^2}{2L^2}}\sqrt{\pi}}{\sqrt{\frac{1}{L^2}}}\right]$$

Out[4]=

$$\frac{1}{2} e^{-\frac{R0^2}{2L^2}} \sqrt{\frac{1}{L^2}}\, L^4 \sqrt{\pi}\, \mathrm{BesselI}\left[1, \frac{R0^2}{2\,L^2}\right]$$

$$U_3 = \int_{-\infty}^{\infty}\int_0^{\infty} r {u_\phi}^2 dr dx = \frac{\theta\sqrt{\pi}}{4\pi R_0^3}[J^2/(2(MR_0)^2)]\theta^2 \exp(-\frac{\theta^2}{2})\mathrm{I}_1(\frac{\theta^2}{2}).$$

Since in this case $\Psi_{VRC} = \Psi_{VR}(\Psi_0 \to 0)$, the translational velocity and kinetic energy for the unbounded swirling vortex ring can be expressed in the closed forms (Fukumoto and Kaplanski, 2010; 2013)

$$U = U_1 + U_3 = \frac{M\theta\sqrt{\pi}}{4\pi^2 R_0^3}[\{3\exp(-\frac{\theta^2}{2})\mathrm{I}_1(\frac{\theta^2}{2}) + \frac{1}{12}\theta^2\ {}_2\mathrm{F}_2\left(\{\frac{3}{2},\frac{3}{2}\},\{\frac{5}{2},3\},-\theta^2\right.$$

$$-\frac{3}{5}\theta^2\ {}_2\mathrm{F}_2\left(\{\frac{3}{2},\frac{5}{2}\},\{2,\frac{7}{2}\},-\theta^2\right)\} - \frac{\Omega}{2}\theta^2\exp(-\frac{\theta^2}{2})\mathrm{I}_1(\frac{\theta^2}{2})], \qquad (33)$$

$$E = E_1 + E_3 = \frac{M^2\theta\sqrt{\pi}}{2\pi^2 R_0^3}[\frac{1}{12}{}_2\mathrm{F}_2\left(\{\frac{3}{2},\frac{3}{2}\},\{\frac{5}{2},3\},-\theta^2\right) + \frac{\Omega}{2}\theta^2\exp(-\frac{\theta^2}{2})\mathrm{I}_1(\frac{\theta^2}{2} \qquad (34)$$

where $\Omega = J^2/(MR_0)^2$.

Comparison with numerical simulation

$\Gamma/\nu = 150$ $\Gamma/\nu = 800$

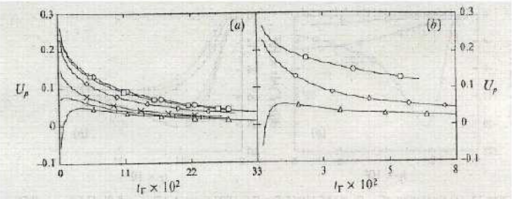
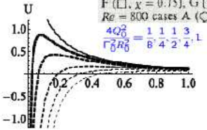


FIGURE 17. (a) Evolution of propagation velocity defined in (31) for $Re = 150$ cases B (○, $\chi = 1.0$), F (□, $\chi = 0.75$), G (◇, $\chi = 0.5$), H (×, $\chi = 0.25$), I (+, $\chi = 0.125$), J (△, $\chi = 0$). (b) The same for $Re = 800$ cases A (○, $\chi = 1.0$), D (◇, $\chi = 0.5$) and E (△, $\chi = 0$).



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Plots of propagation speed of a viscous vortex ring against time for different swirls.

Swirling vortex rings: Comparison with previous numerical simulations

Taken from Cheng, Lou and Lim, 2010

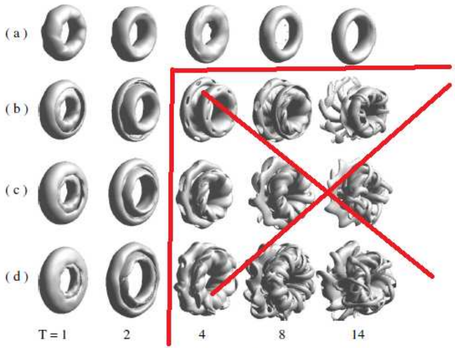


FIG. 7. Transitional vortex structure visualized by isosurface of vorticity for Re=800 and C=0.4 at different swirl numbers. (a) S=0.4; (b) S=0.8; (c) S=1.2; (d) S=2.0.

Swirling vortex rings: Data consistency

From time scale in (Cheng, Lou and Lim, 2010)

$T = \frac{\Gamma_0}{\nu}\frac{\nu t}{{R_0}^2} = 800\frac{L_1^2}{2{R_0}^2} = 800\frac{1}{2{\theta_1}^2}$

Comparison of the initial distributios ω allows to find $\theta_1 = 10\theta$ and $T = 40\frac{1}{\theta^2}$

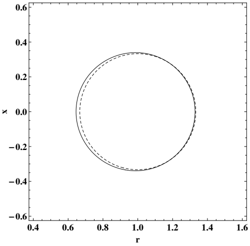
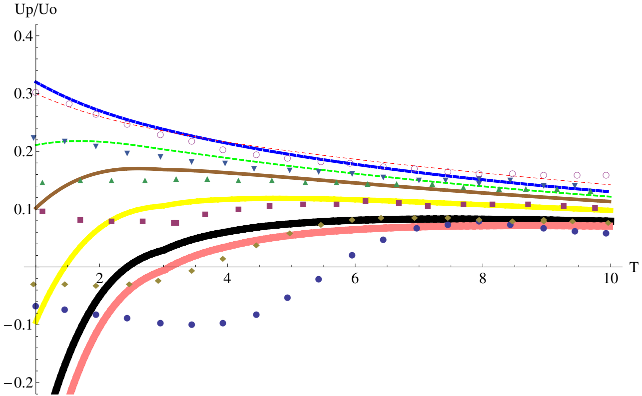


Figure: Comparison of the initial vorticity distributions for $\theta_0 = \sqrt{40}$ with taken from (Cheng, Lou and Lim, 2010) for core size C=0.4 at T=1.

Swirling vortex rings: Comparison with previous numerical simulations



Plots of propagation speed of a viscous vortex ring against time for different core sizes and swirls. Present results for unbounded swirling ring versus data by (Cheng, Lou and Lim, 2010)

Energy, enstrophy, helicity

$$E = \frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}V$$
$$= \frac{\sqrt{\pi}\Gamma_0^2 R_0^4}{48\sqrt{2}(\nu t)^{3/2}}\left\{ {}_2F_2\left(\frac{3}{2},\frac{3}{2};\frac{5}{2},3;-\frac{R_0^2}{2\nu t}\right) + \frac{48 Q_0^2 R_1^4}{\Gamma_0^2 R_0^4 \nu t} e^{-\frac{R_1^2}{4\nu t}} I_1\left(\frac{R_1^2}{4\nu t}\right)\right\}$$

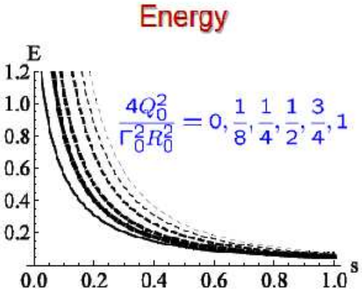
$$Z = \int \omega^2 \mathrm{d}V$$
$$= \frac{\sqrt{\pi}\Gamma_0^2 R_0^2}{4\sqrt{2}(\nu t)^{3/2}}\left\{ e^{-\frac{R_0^2}{4\nu t}} I_1\left(\frac{R_0^2}{4\nu t}\right) + \frac{Q_0^2 R_1^4}{\Gamma_0^2 R_0^4 \nu t}\int_0^\infty \left[\frac{1}{\sigma}\frac{\partial(\sigma W)}{\partial \sigma}\right]^2 \sigma \mathrm{d}\sigma + \frac{1}{2} e^{-\frac{R_1^2}{4\nu t}} I_1\left(\frac{R_1^2}{4\nu t}\right)\right\};$$

where $W = \exp\left(-\frac{\sigma^2}{2} - \frac{R_1^2}{4\nu t}\right) I_1\left(\frac{R_1 \sigma}{\sqrt{2\nu t}}\right)$

$$\mathcal{H} = \int \boldsymbol{u}\cdot\boldsymbol{\omega}\,\mathrm{d}V$$
$$= \frac{\sqrt{2\pi}\Gamma_0^2 Q_0 R_0 R_1}{4(\nu t)^{3/2}} e^{-\frac{R_0^2+R_1^2}{4\nu t}} \int_0^\infty e^{-\sigma^2} I_1\left(\frac{R_0\sigma}{\sqrt{2\nu t}}\right) I_1\left(\frac{R_1\sigma}{\sqrt{2\nu t}}\right)\sigma \mathrm{d}\sigma$$

Energy and helicity

$R_0 = R_1$



$$s = \frac{\nu t}{R_0^2}$$

Concluding Remarks

- It is shown that the earlier obtained Stokes solution of the Navier - Stokes equations can be useful for initializing swirling vortex ring in bounded flow.
- Suggestion: complement the considered results by numerical simulations

Acknowledgements

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