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Non-modal stability of round viscous jets

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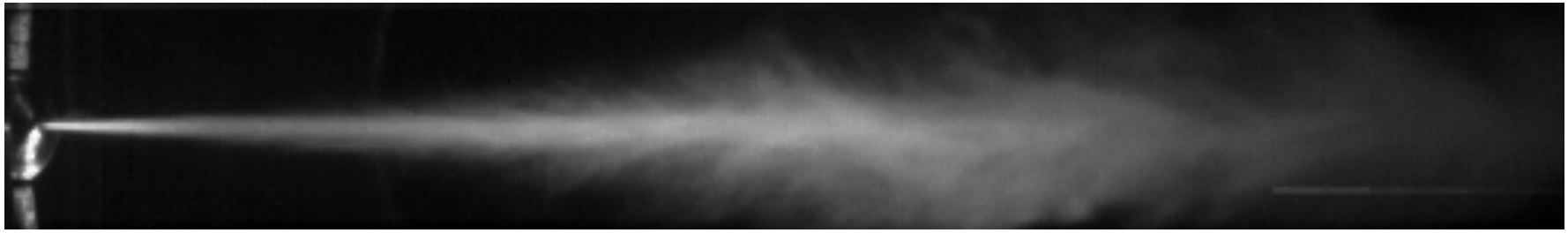
Outline

- Motivation, key ideas of modal and non-modal stability analisys
- Governing equations and boundary conditions
- Main flow velocity profiles
- Results and discussion
- Summary and conclusions

Motivation

Diesel jet break-up in internal combustion engines



- <u>Aim of the study</u>:

  improvement of theoretical predictions of break-up lengths for round viscous jets
- Stability of jets was analysed previously only within the framework of classical modal approach. Non-modal instability?

Key ideas of modal stability approach

Main plane-parallel shear flow:

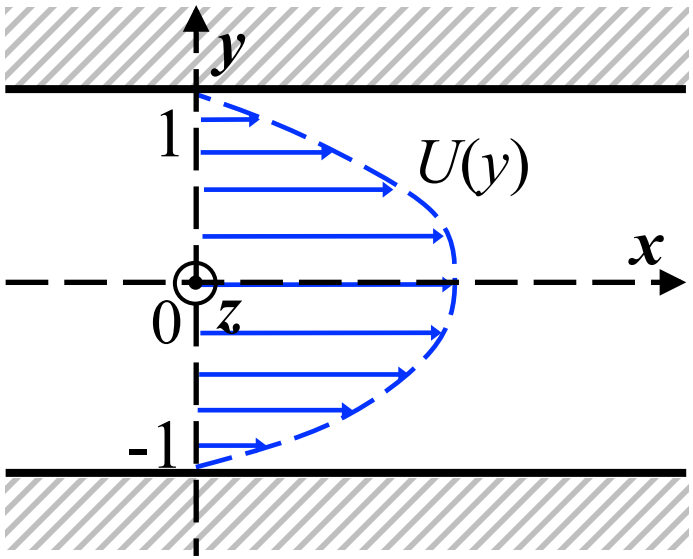
$$\mathbf{V} = \{U(y), 0, 0\}$$

Small disturbances:

$$\mathbf{v} = \mathbf{V} + \mathbf{v}', \quad p = P + p'$$

Consider solutions in the form of travelling waves (equivalent to Fourier-Laplace transform):

$$\mathbf{q}'(\mathbf{r}, t) = \mathbf{q}(y)\exp(i(k_x x + k_z z - \omega t))$$



Eigenvalue problem:

$$\begin{cases} L\mathbf{q} = i\omega\mathbf{q}, \\ \mathbf{q}(y = \pm 1, t) = 0 \end{cases}$$

$\mathbf{q}$ – vector of independent variables (normal velocity and normal vorticity for 3D disturbances), L – linear ordinary differential operator

Solution is the system of eigenfunctions (normal modes): $\{\mathbf{q}_n(\mathrm{y}), \omega_n(k_x, k_z)\}$

The flow is stable if all normal modes decay $(\mathrm{Im}\{\omega_n\}<0, \forall n)$

Modal stability: pros and cons

- ✓ Squire theorem (**2D disturbances** are the most unstable)
- ✓ **Modal theory predicts values of critical Reynolds numbers** for several shear flows (plane channel, boundary layer)
- ✓ **Examples of failures**: Poiseuille pipe flow (stable at any Re according to modal theory, unstable in experiments!)
- ✓ Transition of shear flows is usually accompanied by **3D streamwise-alongated disturbances** (”streaks”, see Fig.)

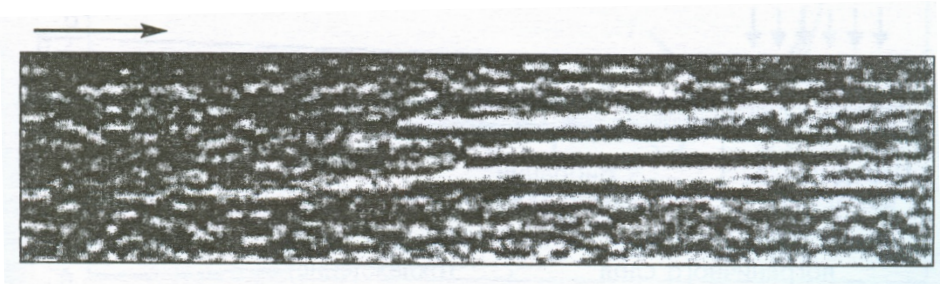


Fig. Visualization of streaks in boundary-layer flow[1]

[1] Alfredsson P.H., Bakchinov A.A., Kozlov V.V., Matsubara M. Laminar-Turbulent transition at a high level of a free stream turbulence. In: Nonlinear instability and transition in three-dimenasional boundary layers Eds. P.H. Duck, P. Hall. Dordrecht, Kluwer, pp. 423-436 (1996)

Algebraic (non-modal) instability

- A necessity for <u>linear</u> "bypass transition" theories (non-modal growth)
- Mathematical reason for non-modal instability:
  - Differential operators are non-Hermitian (**eigenvectors are not orthogonal**)
  - Solutions are linear combinations of normal modes, **growth is possible even if all modes decay** (see Fig.)

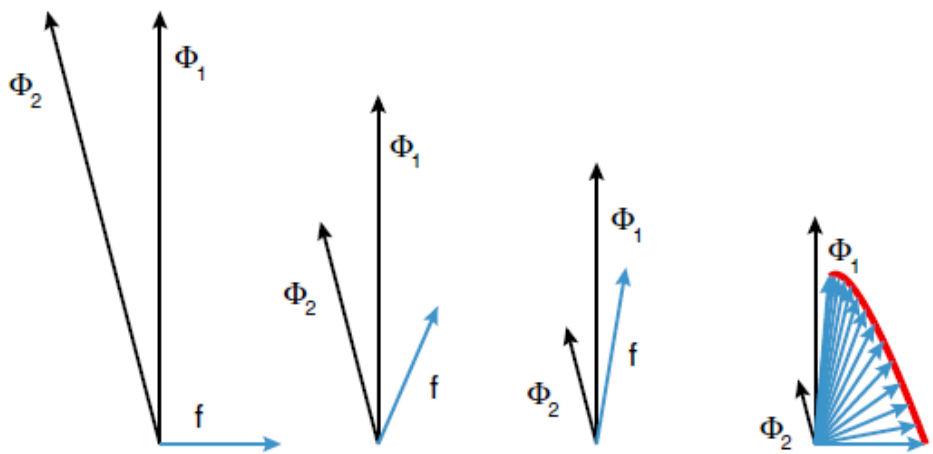


Fig. Time-evolution of the difference of two decaying non-orthogonal vectors[2]

[2] P.J. Schmid, Nonmodal Stability Theory, Annu. Rev. Fluid Mech, V. 39, pp. 129-162 (2007)

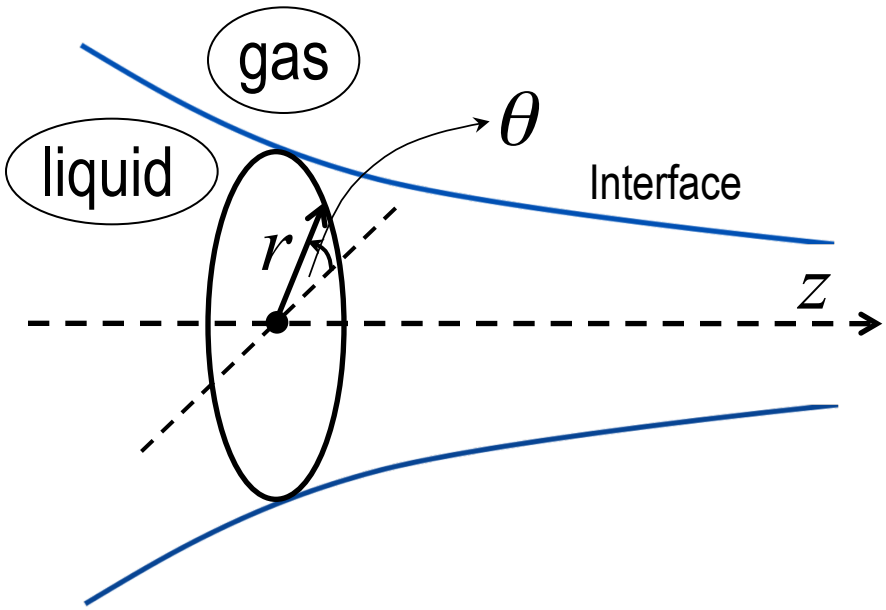
Flow configuration

- Surrounding gas and jet liquid are incompressible and viscous Newtonian fluids

- Cylindrical coordinate system (z, r, θ)

- Parameters of fluids (gas and liquid, $\alpha = g, l$)

ρ_α, μ_α are densities and viscosities
$\mathbf{v}_\alpha$, p_α are velocities and pressures



Non-dimensional governing equations

$$\frac{\partial u_\alpha}{\partial z}+\frac{1}{r}\frac{\partial v_\alpha r}{\partial r}+\frac{1}{r}\frac{\partial w_\alpha}{\partial \theta}=0$$

$$\frac{\partial u_\alpha}{\partial t}+u_\alpha\frac{\partial u_\alpha}{\partial z}+v_\alpha\frac{\partial u_\alpha}{\partial r}+\frac{w_\alpha}{r}\frac{\partial u_\alpha}{\partial \theta}=-\frac{\partial p_\alpha}{\partial z}+\frac{1}{\mathrm{Re}_\alpha}\Delta u_\alpha$$

$$\frac{\partial v_\alpha}{\partial t}+u_\alpha\frac{\partial v_\alpha}{\partial z}+v_\alpha\frac{\partial v_\alpha}{\partial r}+\frac{w_\alpha}{r}\frac{\partial v_\alpha}{\partial \theta}-\frac{w_\alpha^2}{r}=-\frac{\partial p_\alpha}{\partial r}+\frac{1}{\mathrm{Re}_\alpha}\left(\Delta v_\alpha-\frac{2}{r^2}\frac{\partial w_\alpha}{\partial \theta}-\frac{v_\alpha}{r^2}\right)$$

$$\frac{\partial w_\alpha}{\partial t}+u_\alpha\frac{\partial w_\alpha}{\partial z}+v_\alpha\frac{\partial w_\alpha}{\partial r}+\frac{w_\alpha}{r}\frac{\partial w_\alpha}{\partial \theta}+\frac{v_\alpha w_\alpha}{r}=-\frac{1}{r}\frac{\partial p_\alpha}{\partial \theta}+\frac{1}{\mathrm{Re}_\alpha}\left(\Delta w_\alpha+\frac{2}{r^2}\frac{\partial v_\alpha}{\partial \theta}-\frac{w_\alpha}{r^2}\right)$$

$$\alpha=l,g,\quad \Delta=\frac{\partial^2}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},\quad \mathrm{Re}_l=\frac{UL\rho_l}{\mu_l},\quad \mathrm{Re}_g=\mathrm{Re}_l\frac{\eta}{\zeta},\quad \eta=\frac{\rho_g}{\rho_l},\quad \zeta=\frac{\mu_g}{\mu_l}$$

Flow scales:

U – liquid velocity at the axis;
L – jet radius;
$\rho_l U^2$ – liquid pressure p_l;
$\rho_g U^2$ – gas pressure p_g;

Independent governing parameters:

Re_l – Reynolds number for jet liquid,

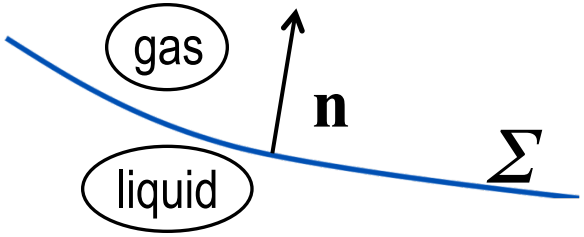
η, ζ – gas-to-liquid density and viscosity ratios

Boundary conditions

Interface $\boldsymbol{\Sigma}$: $H = r - h(z,\theta,t) = 0$

Normal unit vector $\mathbf{n}$: $\mathbf{n} = \nabla H / |\nabla H|$, $\nabla H = \left\{ -\frac{\partial h}{\partial z}, 1, -\frac{1}{r}\frac{\partial h}{\partial \theta} \right\}$

- Conditions at the interface:



Kinematic condition: $\frac{dH}{dt} = 0 \Leftrightarrow \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial z} - v = 0$

Continuity of velocity (no-slip): $u_l = u_g$, $v_l = v_g$, $w_l = w_g$

Jump in stress due to capillary force $\mathbf{R}$: $\mathbf{s}_l - \eta \mathbf{s}_g = \mathbf{R}$

$$\mathbf{s}_\alpha = -p_\alpha n_i + \frac{1}{\mathrm{Re}_\alpha}\left(\nabla^i v^j + \nabla^j v^i\right) n_j; \quad R_i = \frac{1}{\mathrm{We}} n_i \operatorname{div} \mathbf{n}, \quad \boxed{\mathrm{We} = \frac{\rho_l U^2 L}{\sigma};}$$

- Kinematic condition at the axis[3] ($r \to 0$): $\frac{\partial u_l}{\partial \theta}, \frac{\partial p_l}{\partial \theta} \to 0, \quad v_l, w_l \sim \sin\theta$
- At the infinity ($r \to \infty$): $u_g, v_g, w_g, p_g \to 0$

[3] G.K. Batchelor, A.E. Gill, Analysis of the stability of axisymmetric jets. J. Fluid. Mech., V.14, pp. 529-551 (1962)

Main flow: immersed jet

Two velocity profiles for immersed jet (single fluid, no interface):

- "Top-hat" profile (close to the orifice)

$$U_1(r) = \begin{cases} 1, & r \le 1 \\ \exp\left\{-(r-1)^2/\delta^2\right\}, & r > 1 \end{cases}$$

- "Smooth" profile (far downstream)[4]

$$U_2(r) = \frac{1}{\left(1+r^2\right)^2}$$

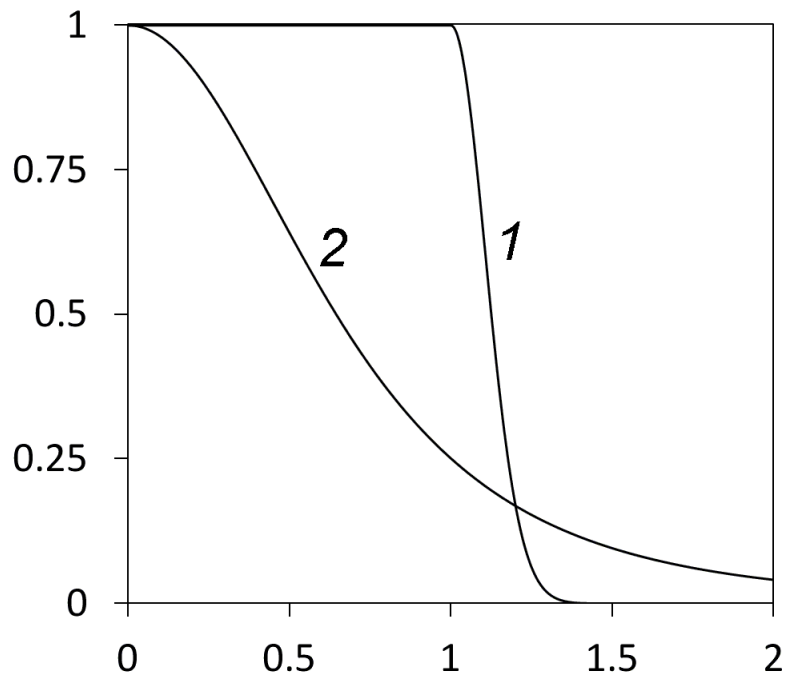


Fig. Velocity profiles of the immersed jet considered

[4] L.D. Landau, E.M. Lifshitz, *Fluid Mechanics,* Pergamon, 1959.

Main flow: jet in the air

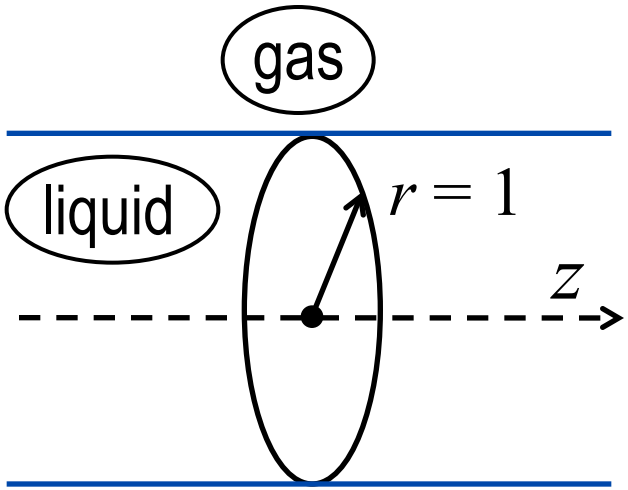
- "Local" axisymmetric velocity profile (cylindrical jet of the fixed radius $r = 1$):

$$\mathbf{V}_{l,g} = \{U_{l,g}(r), 0, 0\},$$

$$r \to \infty: \quad U_g(r) \to 0,$$

$$r = 1: \quad U_l = U_g, \quad U_l' = \zeta U_g',$$

$$r = 0: \quad |U_l|, |P_l| < \infty,$$



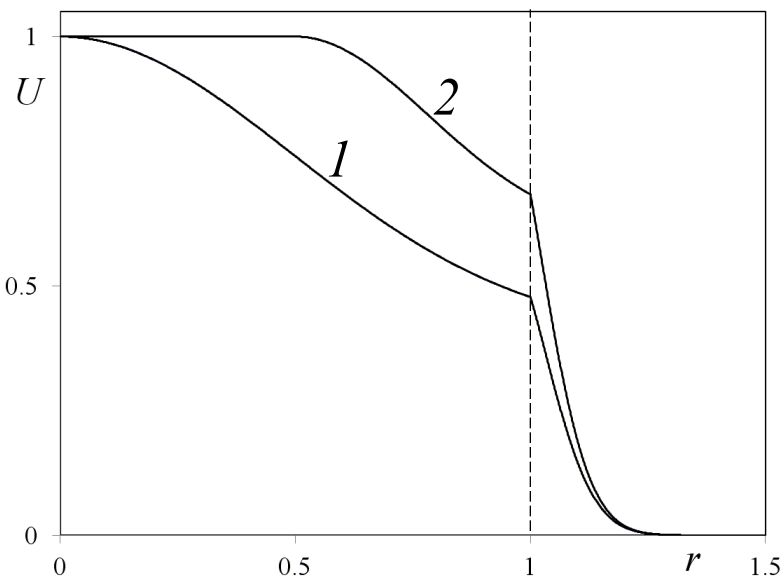


Fig. Jet in the air at ζ=0.1, "model" velocity profiles 1 and 2 considered (liquid at r<1 and gas at r>1)

Linear stability problem

Linearized Navier-Stokes equations for each fluid ($\alpha = l, g$):

$$\begin{cases} \dfrac{\partial u_\alpha}{\partial z} + \dfrac{1}{r}\dfrac{\partial v_\alpha r}{\partial r} + \dfrac{1}{r}\dfrac{\partial w_\alpha}{\partial \theta} = 0 \\ \dfrac{\partial u_\alpha}{\partial t} + U_\alpha \dfrac{\partial u_\alpha}{\partial z} + v_\alpha U'_\alpha = -\dfrac{\partial p_\alpha}{\partial z} + \dfrac{1}{\mathrm{Re}_\alpha}\left(\Delta u_\alpha\right) \\ \dfrac{\partial v_\alpha}{\partial t} + U_\alpha \dfrac{\partial v_\alpha}{\partial z} = -\dfrac{\partial p_\alpha}{\partial r} + \dfrac{1}{\mathrm{Re}_\alpha}\left(\Delta v_\alpha - \dfrac{2}{r^2}\dfrac{\partial w_\alpha}{\partial \theta} - \dfrac{v_\alpha}{r^2}\right) \\ \dfrac{\partial w_\alpha}{\partial t} + U_\alpha \dfrac{\partial w_\alpha}{\partial z} = -\dfrac{1}{r}\dfrac{\partial p_\alpha}{\partial \theta} + \dfrac{1}{\mathrm{Re}_\alpha}\left(\Delta w_\alpha + \dfrac{2}{r^2}\dfrac{\partial v_\alpha}{\partial \theta} - \dfrac{w_\alpha}{r^2}\right) \end{cases}$$

$$\Delta = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \quad \mathrm{Re}_g = \mathrm{Re}_l \frac{\eta}{\zeta},$$

Linearized boundary conditions

Conditions at the interface at $r = 1$ ($h << 1$ is the disturbance of the interface)

- Continuity $\quad u_l + hU_l' = u_g + hU_g', \quad v_l = v_g, \quad w_l = w_g$
- Kinematic condition $\quad \dfrac{\partial h}{\partial t} + U_l \dfrac{\partial h}{\partial x} = v_l$
- Force balance

$$\frac{\partial u_l}{\partial y} + \frac{\partial v_l}{\partial z} + hU_l'' = \zeta\left(\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial z} + hU_g''\right),$$

$$p_l - \frac{2}{\mathrm{Re}_l}\frac{\partial v_l}{\partial y} = \eta p_g - \frac{2\zeta}{\mathrm{Re}_l}\frac{\partial v_g}{\partial y} - \frac{1}{We}\left(\frac{\partial^2 h}{\partial^2 \theta} + \frac{\partial^2 h}{\partial^2 z} + h\right)$$

$$\frac{\partial w_l}{\partial y} + \frac{\partial w_l}{\partial \theta} + w_l = \zeta\left(\frac{\partial w_g}{\partial y} + \frac{\partial w_g}{\partial \theta} + w_g\right)$$

- Kinematic condition at the axis ($r \to 0$): $\quad \dfrac{\partial u_l}{\partial \theta}, \dfrac{\partial p_l}{\partial \theta} \to 0, \quad v_l, w_l \sim \sin\theta$
- At the infinity ($r \to \infty$): $\quad u_g, v_g, w_g, p_g \to 0$

Spatial normal modes and eigenvalue problem

Travelling waves (normal modes growing in *z*):

$$\mathbf{q}(z,r,\theta,t) = \mathbf{q}_*(r)\exp\{i(kz + m\theta - \omega t)\}, \quad \mathbf{q} = \{u, v, w, p\}$$

(k – complex wave number, m – integer azimuthal number, ω - real frequency)

- Eigenvalue problem:

$$\boxed{L\mathbf{q} = 0}$$ (L – second order linear differential operator 4x4)

$$\left.\begin{aligned} r = 0: &\quad A\mathbf{q} = 0 \\ r = 1: &\quad B\mathbf{q} = 0 \\ r \to \infty: &\quad C\mathbf{q} = 0 \end{aligned}\right\}$$ homogeneous boundary conditions

- System of eigenvalues and eigenfunctions:

$$\forall \mathrm{Re}_l, \xi, \eta, \mathrm{We}, \omega, m: \ \{\mathbf{A}_n, k_n\}, \ n = 1,2,\ldots$$ (+ continuum part of the spectrum)

- Modal stability analysis: flow is stable if $\mathrm{Im}\{k_n\} > 0$, $\forall n$

Non-modal stability: optimal disturbances

- Expanding the disturbance into series of eigenfunctions at given m, ω:

$$\mathbf{q}(z,r,\theta,t) = \left(\sum_{n=1}^{\infty} \gamma_n \mathbf{q}_n(r) \exp(ik_n z) \right) \exp\{i(m\theta - \omega t)\}$$

(the set of coefficients γ_n is a spectral projection of a disturbance **q**)

- Evaluation of the growth: density of the kinetic energy

$$E(t,\gamma) = \frac{1}{2ab} \int_0^a \int_0^b \int_0^R \left(\mathrm{Re}\{u\}^2 + \mathrm{Re}\{v\}^2 + \mathrm{Re}\{w\}^2 \right) r\,dr\,d\theta\,dt$$

$$a = 2\pi / m, \;\; b = 2\pi / \omega, \quad R >> 1$$

- Disturbance with maximum energy at a given position z (optimal disturbances):

$$\gamma - ?: \quad E(t,\gamma) \to \max_{\gamma}, \quad E(0,\gamma) = 1$$

Algorithm of the numerical solution

- **Variable mapping** (non-uniform mesh, refinement with a decrease in *r*)

- **System of *N* spatial normal modes is found by orthonormalization method**[5]
  $\forall \mathrm{Re}_l, \eta, \xi, \mathrm{We}, \omega, m: \ \{\mathbf{q}_n(r), k_n\}, \ n = 1 \ldots N$

- **Optimal perturbations are found** by Lagrange-multiplier technique
  (Resulting generalized eigenvalue problem for energy matrices is solved by QR-algorithm)

[5] S.K. Godunov, On the numerical solution of boundary-value problems for systems of linear differential equations [in Russian], Uspekhi mat. Nauk, V.16, Iss.3, pp.171-174 (1961).

Results: *Immersed jet, "top-hat" velocity profile*

Non-modal stability of jet with "top-hat" velocity profile (close to the orifice)

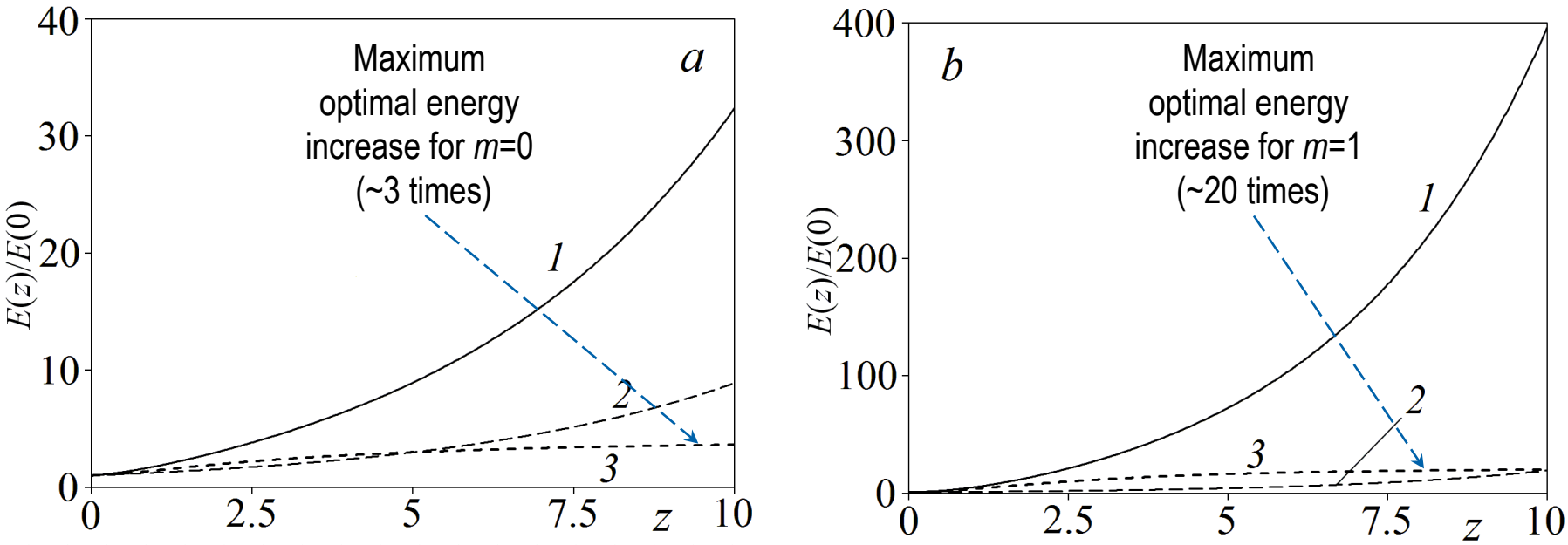


Fig. Normalized kinetic energy of the optimal disturbance for 'top-hat' velocity profile Vs. position downstream z for ω = 0.5, m=0 (a) and ω = 0.2, m=1 (b), Re = 1000. *1* – optimal energy, *2* – energy of the first mode only, *3* – ratio of optimal to single-mode energies

- "Top-hat" velocity profile is unstable for all values of azimuthal number m (first mode grows)
- Highest optimal-to-single mode energy ratio is 20, it corresponds to m=1 (Re=1000)

Results: *Immersed jet, "smooth" velocity profile*

Non-modal stability of jet with "smooth" velocity profile (far downstream)

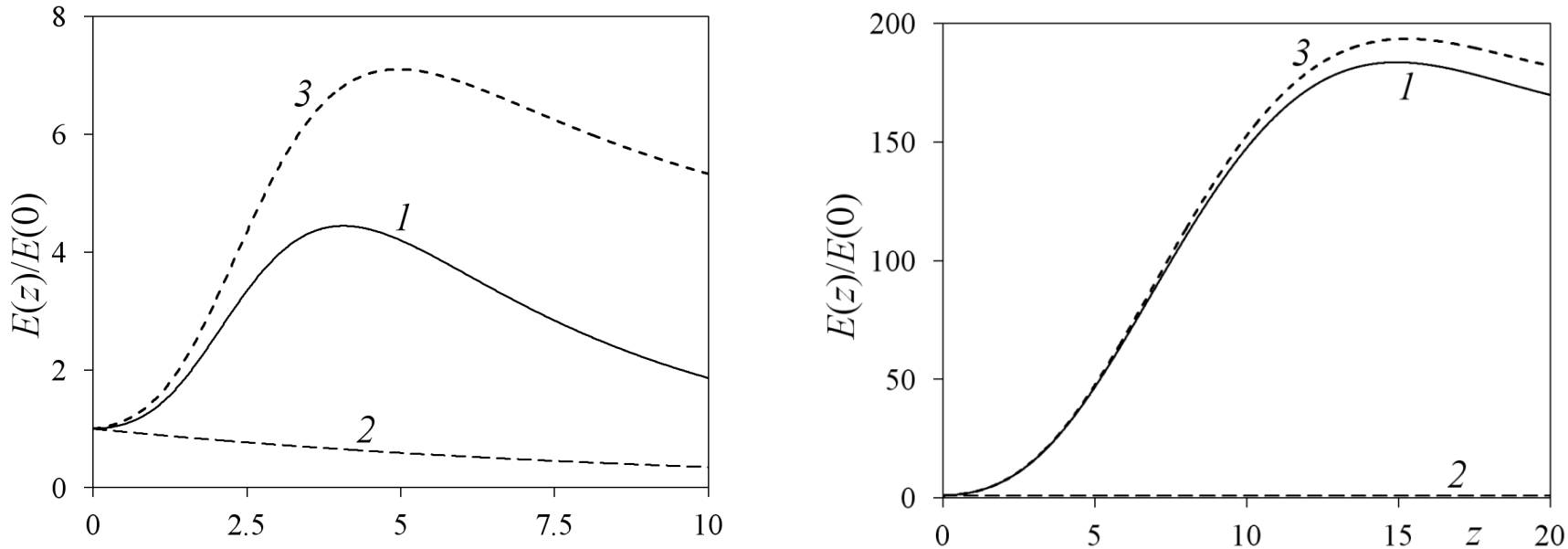


Fig. Normalized kinetic energy of optimal disturbance for 'smooth' profile Vs. position downstream *z* for ω = 0.5, *m*=0 (*a*) and ω = 0.2, *m*=1 (*b*), Re = 1000. *1* – optimal energy, *2* – energy of the first mode only, *3* – ratio of the optimal to single-mode energies.

- Smooth velocity profile is stable at *m*=0 and unstable at *m*>0 (P.J. Morris, 1976)
- Non-modal growth is significantly stronger for smooth profile and *m*>0 (optimal-to-single mode energy ratio around 200 and corresponds to *z*~15)

Results: *jet in the air, velocity profile 1*

Non-modal stability of jet in the air (Fig. 1), velocity profile is shown in Fig. 2.

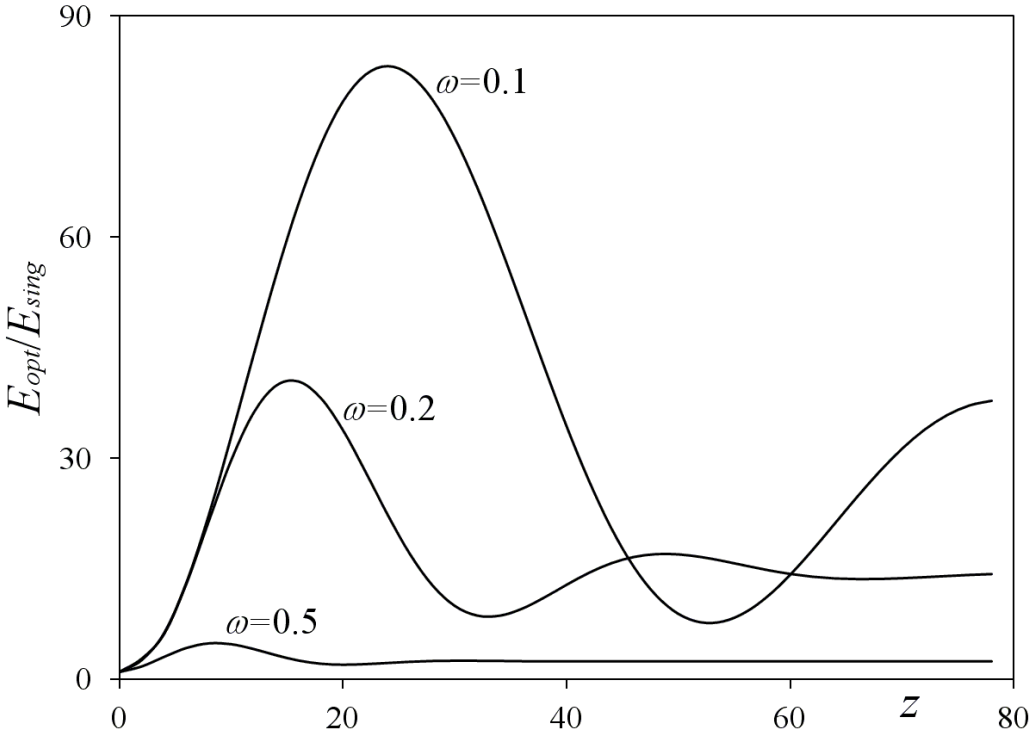


Fig. 1. Optimal-to-single mode energy ratio for jet in the air with velocity profile 1 (Fig. 2) Vs. position downstream z for m=1, η=0.001, ζ=0.01, Re = 1000. Curves correspond to ω =0.1, 0.2, 0.5

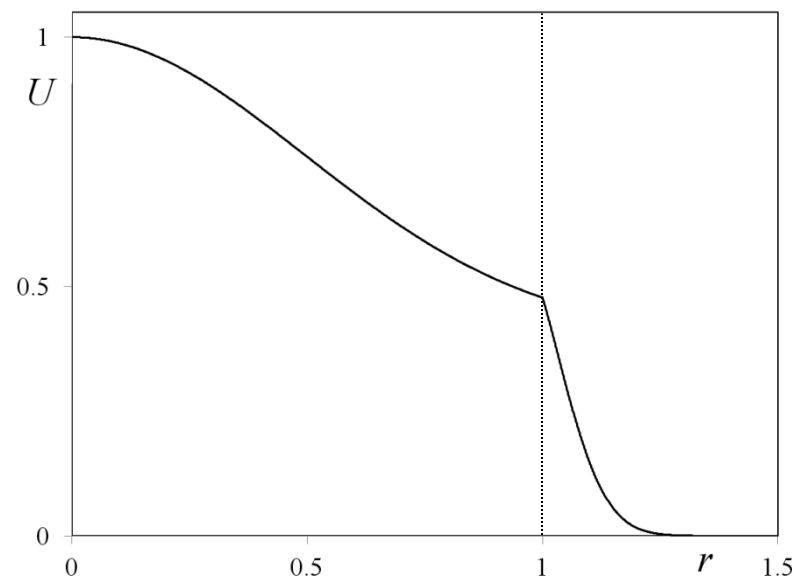


Fig. 2. Main flow velocity profile 1

- Non-modal growth decreases with an increase in frequency ω of the disturbance
- Highest optimal growth corresponds to non-axisymmetric disturbances (m>0)
- Highest optimal-to-single mode energy ratio is of order of 10^2 for ω=0.1

Results: *jet in the air, velocity profile 2*

Non-modal stability of jet in the air (Fig. 1), velocity profile is shown in Fig. 2.

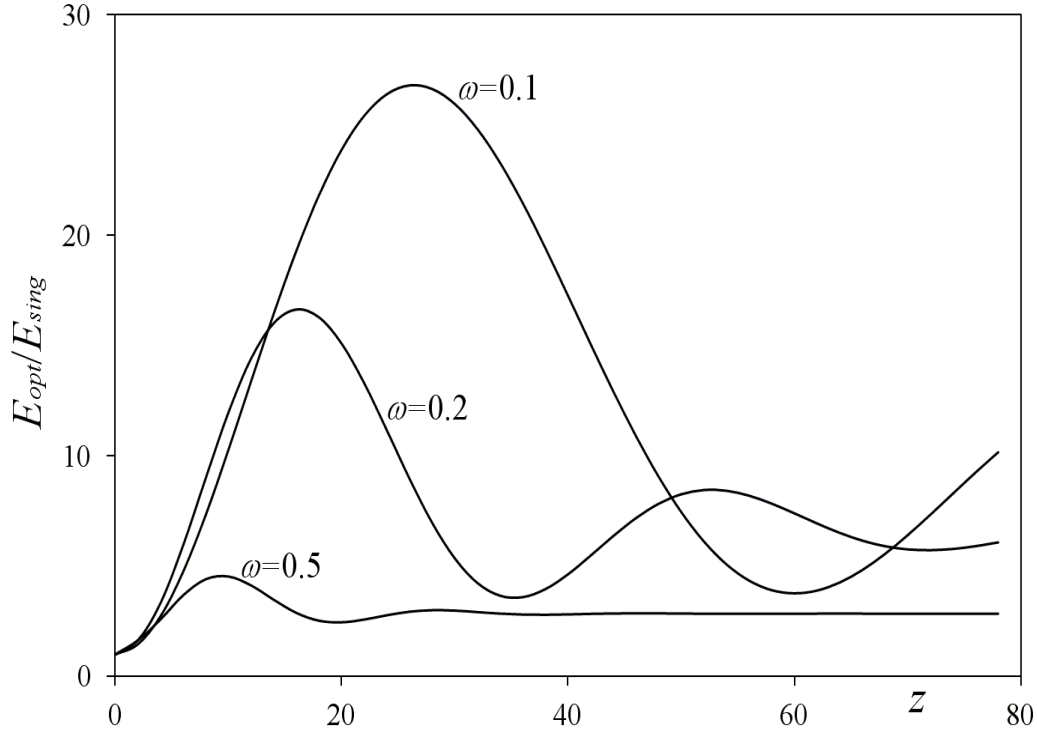


Fig. 1. Optimal-to-single mode energy ratio for jet in the air with velocity profile 2 (Fig. 2) Vs. position downstream z for m=1, η=0.001, ζ=0.01, Re = 1000. Curves correspond to ω =0.1, 0.2, 0.5

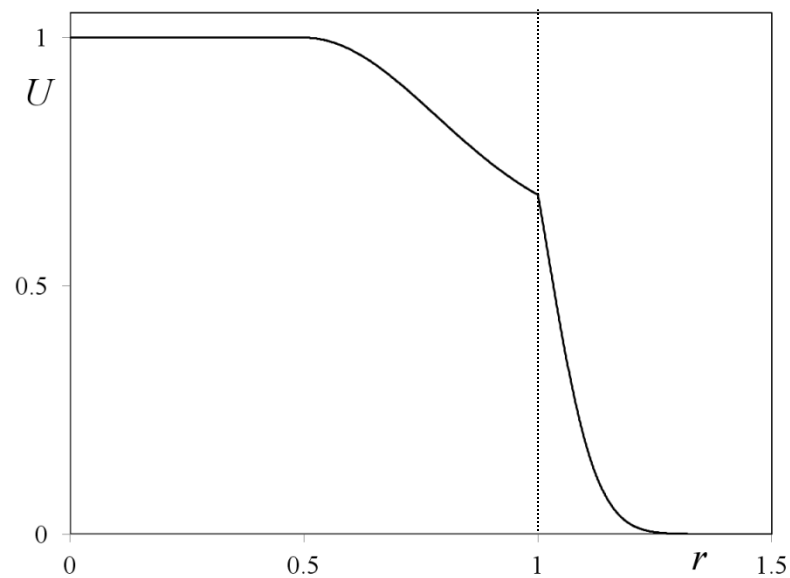


Fig. 2. Main flow velocity profile 2

Summary and conclusions

- Linear stability problem for spatially-developing disturbances in round viscous jet is formulated. The effects of surface tension and ambient gas are considered
- Numerical algorithm for modal and non-modal stability study is developed and validated
- Parametric study of optimal spatial disturbances to both immersed jet and jet in the air is carried out
- For the case of both jet flows, it is found that non-modal instability mechanism is strongest for non-axisymmetric disturbances (m>0) and it is damped with an increase in frequency of the disturbances. Maximum optimal-to-single mode energy ratio is of order of 10^2

Thank you for attention!

Energy norm and optimal spatial disturbances

$$\mathbf{q}(r,z,\theta,t) = \left(\sum_{n=1}^{N} \gamma_n \mathbf{q}_n(r) \exp(ik_n z) \right) \exp\{i(m\theta - \omega t)\}$$

- **Energy norm:**

$$E(\gamma, z) = \int \left(uu^* + vv^* + ww^* \right) r dr = \boldsymbol{\gamma}^* E_z \boldsymbol{\gamma}$$

$$E_z = \mathbf{U}^* \mathbf{U} + \mathbf{V}^* \mathbf{V} + \mathbf{W}^* \mathbf{W}$$

- **Maximization of energy functional:**

$\boldsymbol{\gamma} - ?: \quad \boldsymbol{\gamma}^* E_z \boldsymbol{\gamma} \to \max_{\gamma}, \quad \boldsymbol{\gamma}^* E_0 \boldsymbol{\gamma} = 1$ (E_z is positive Hermitian quadratic form)

Euler-Lagrange equations: $\boxed{E_z + \sigma E_0 = 0}$

(generalized eigenvalue problem for energy matrix)

Optimal disturbances correspond to eigenvector with highest eigenvalue σ