Analytical methods in liquid breakup modelling

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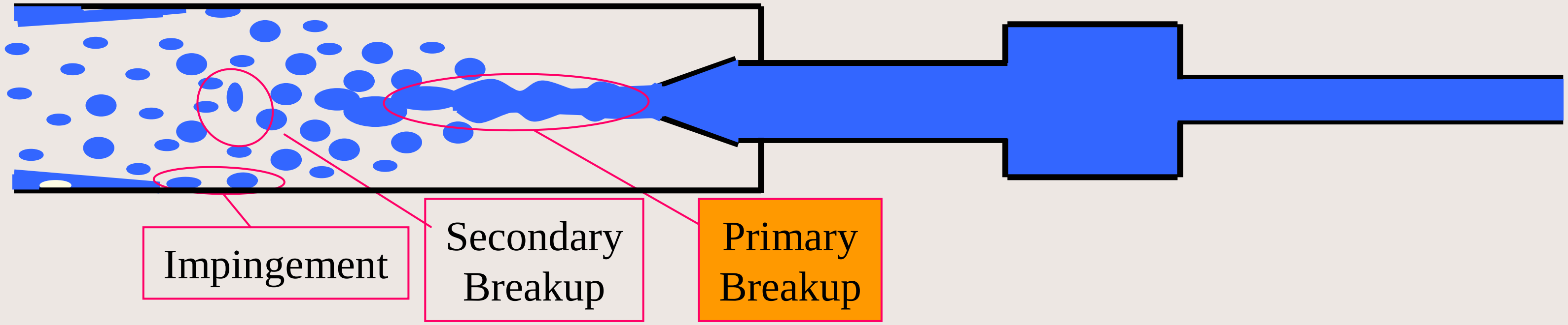
Overview



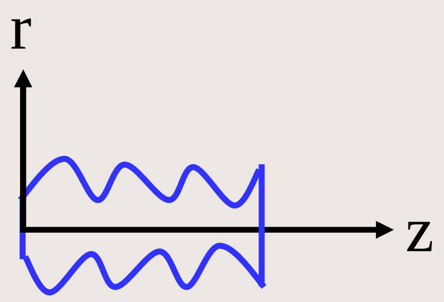
Basic equations

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta)+\frac{\partial}{\partial z}(u_z)=0$$

z - Momentum:

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\nabla^2 u_z$$



r - Momentum:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\nabla^2 u_r$$

Instability perturbations:

$$u_z=\bar{u}_z+u_z' \qquad u_r=\bar{u}_r+u_r' \qquad p=\bar{p}+p' \qquad R=R_0+\delta$$

Perturbations form

For a transient inviscid jet, where: $\bar{u}_z(t)$

Continuity: $$\frac{1}{r}\frac{\partial}{\partial r}(r u'_r) + \frac{\partial}{\partial z}(u'_z) = 0$$

$$\frac{\partial \bar{u}_z}{\partial z} = \frac{1}{\bar{u}_z}\frac{\partial \bar{u}_z}{\partial t} \to 0$$

z-Momentum: $$\rho\left(\frac{\partial u'_z}{\partial t} + \bar{u}_z\frac{\partial u'_z}{\partial z} + \cancel{u'_z\frac{\partial \bar{u}_z}{\partial z}}\right) = -\frac{\partial p'}{\partial z}$$

r-Momentum: $$\rho\frac{\partial u'_r}{\partial t} = -\frac{\partial p'}{\partial r}$$

This analysis:

$$\boxed{\begin{aligned} \delta &= Ae^{\beta(t)+i\alpha z} \\ p' &= f(r)\hat{p}(t)e^{\beta(t)+i\alpha z} \\ u'_z &= f(r)\hat{u}(t)e^{\beta(t)+i\alpha z} \end{aligned}}$$

Classical analysis:

$$\delta = Ae^{\beta\cdot t+i\alpha z}$$

$$p' = f(r)\hat{p}e^{\beta\cdot t+i\alpha z}$$

$$u'_z = f(r)\hat{u}e^{\beta\cdot t+i\alpha z}$$

Pressure equation

Differentiating the momentum equations and substituting in the continuity:

$$\rho \bar{u}_z \frac{\partial^2 u_z'}{\partial z^2} + \frac{\partial^2 p'}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial p'}{\partial r} \right) = 0$$

Substituting perturbation parameters:

$$-\left(\rho \frac{\bar{u}_z(t)\hat{u}(t)}{\hat{p}(t)} + 1 \right)\alpha^2 f(r) + \frac{1}{r}\frac{d}{dr}\left(r \frac{df(r)}{dr} \right) = 0$$

For vapour region (outer): $f(r) = cK_0(\alpha_* r)$ $\qquad \alpha_* \equiv \alpha\sqrt{1 + \frac{\rho \bar{u}_z(t)\hat{u}(t)}{\hat{p}(t)}} \approx \alpha$

For liquid region (inner): $f(r) = cI_0(\alpha_* r)$

Radial velocity

Integrating the continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r'\right)+\frac{\partial}{\partial z}\left(u_z'\right)=0 \qquad \text{with:} \qquad u_z' = f\left(r\right)\hat{u}\left(t\right)e^{\beta\left(t\right)+i\alpha z}$$

gives:

$$u_r' = -i\alpha\hat{u}\left(t\right)\frac{1}{r}\int r f\left(r\right)dr \cdot e^{\beta\left(t\right)+i\alpha z}$$

Kinematic Boundary Condition

$$u'_r\Big|_{r=R_0} = \frac{\partial \delta}{\partial t} + \bar{u}_z \frac{\partial \delta}{\partial z}$$

For vapour region (outer):

$$\hat{u} = -\frac{A}{K_1(\alpha R_0)}\left[i\frac{d\beta(t)}{dt} - \alpha \bar{u}_z\right]$$

For liquid region (inner):

$$\hat{u} = -\frac{A}{I_1(\alpha R_0)}\left[i\frac{d\beta(t)}{dt} - \alpha \bar{u}_z\right]$$

Dynamic Boundary Condition

$$\left(p_l' - p_v'\right)\Big|_{r=R_0} = \sigma\left(-\frac{\partial^2\delta}{\partial x^2} + \frac{1-\delta/R_0}{R_0}\right)$$

Substituting perturbation parameters:

$$f\left(R_0\right)\left[\hat{p}_l\left(t\right) - \hat{p}_v\left(t\right)\right] = \sigma A\left(\alpha^2 - \frac{1}{{R_0}^2}\right)$$

The dispersion relation

Subtracting the momentum equations of the vapour from the liquid phase, at the surface (for a moving liquid in ambient gas at rest):

$$\left(\frac{I_0(\alpha R_0)}{I_1(\alpha R_0)}\rho_l+\frac{K_0(\alpha R_0)}{K_1(\alpha R_0)}\rho_v\right)\left[\frac{d^2\beta(t)}{dt^2}+\left(\frac{d\beta(t)}{dt}\right)^2\right]+\frac{I_0(\alpha R_0)}{I_1(\alpha R_0)}\rho_l\alpha\left(2i\bar{u}_z\frac{d\beta(t)}{dt}+i\frac{d\bar{u}_z}{dt}-\alpha\bar{u}_z^{\,2}\right)+\sigma\alpha\left(\alpha^2-\frac{1}{R_0^{\,2}}\right)=0$$

For expected short waves: $\alpha R_0\to\infty \quad\Rightarrow\quad \dfrac{I_1(\alpha R_0)}{I_0(\alpha R_0)}\approx 1,\ \dfrac{K_1(\alpha R_0)}{K_0(\alpha R_0)}\approx 1$

$$(\rho_l+\rho_v)\left[\frac{d^2\beta(t)}{dt^2}+\left(\frac{d\beta(t)}{dt}\right)^2\right]+\rho_l\alpha\left(2i\bar{u}_z\frac{d\beta(t)}{dt}+i\frac{d\bar{u}_z}{dt}-\alpha\bar{u}_z^{\,2}\right)+\sigma\alpha\left(\alpha^2-\frac{1}{R_0^{\,2}}\right)=0$$

Simplified solution – steady jet

For a steady jet: $\frac{d\bar{u}_z}{dt} = 0, \quad \frac{d\beta(t)}{dt} \rightarrow const.$

$$\frac{d\beta(t)}{dt} = \frac{-i\alpha\rho_l\bar{u}_z \pm \sqrt{\alpha\rho_l\rho_v\bar{u}_z^{\,2} - (\rho_l+\rho_v)\alpha\sigma\left(\alpha^2 - \frac{1}{R_0^{\,2}}\right)}}{\rho_l+\rho_v}$$

The dominant wave is expected to be at: $\frac{\partial}{\partial\alpha}\mathrm{Re}\left(\frac{d\beta(t)}{dt}\right) = 0$

$$\alpha = \frac{\bar{u}_z^{\,2} + \sqrt{\bar{u}_z^{\,4} - \left(\frac{\rho_l+\rho_v}{\rho_l\rho_v}\sigma\right)^2 \frac{3}{R_0^{\,2}}}}{3\frac{\rho_l+\rho_v}{\rho_l\rho_v}\sigma}$$

$$\alpha \gg \frac{1}{R_0} \longrightarrow \alpha = \frac{2}{3}\frac{\rho_l\rho_v}{\rho_l+\rho_v}\frac{\bar{u}_z^{\,2}}{\sigma}$$

$$\bar{u}_z = 0 \longrightarrow \alpha = \frac{1}{\sqrt{3}R_0}$$

Simplified solution – transient jet

For transient jets of interest, wave growth is more rapid than velocity change, so for the time until breakup:

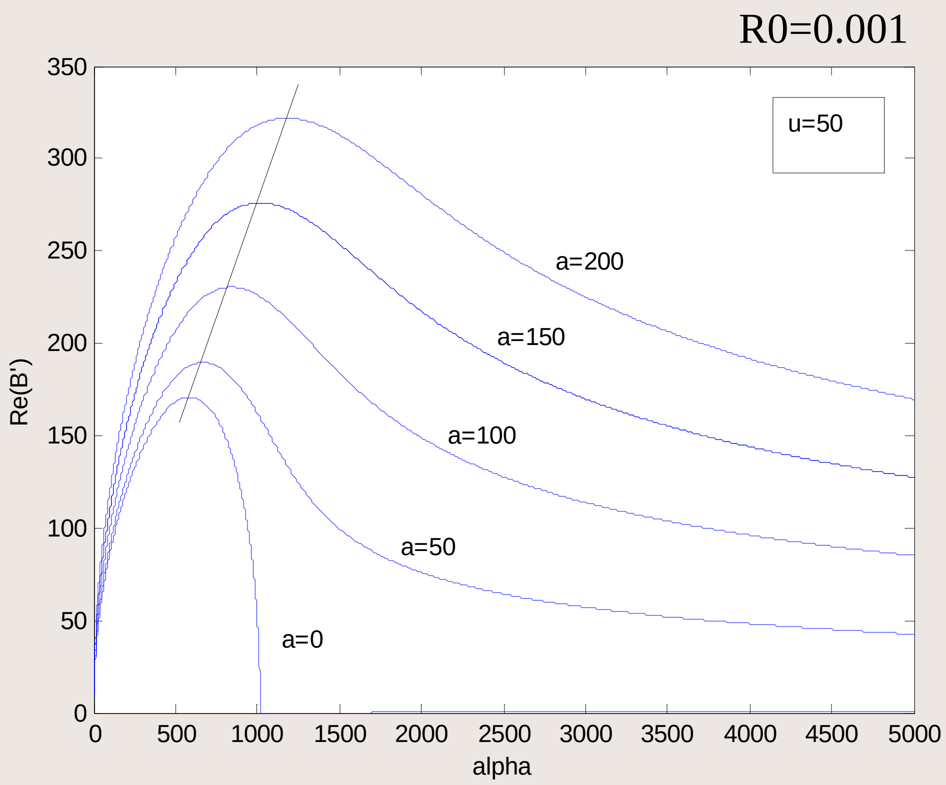
$$\bar{u}_z \approx const., \quad \frac{d\bar{u}_z}{dt} = a, \quad \frac{d\beta(t)}{dt} \to const.$$

$$\frac{d\beta(t)}{dt} = \frac{-i\alpha\rho_l\bar{u}_z \pm \sqrt{\alpha\rho_l\rho_v\bar{u}_z^{\,2} - (\rho_l+\rho_v)\alpha\sigma\left(\alpha^2 - \frac{1}{R_0^{\,2}}\right) - i(\rho_l+\rho_v)\rho_l\alpha a}}{\rho_l+\rho_v}$$

$$\operatorname{Re}\left(\frac{d\beta(t)}{dt}\right) = \frac{\sqrt{\alpha}\sqrt{\sqrt{\left[\rho_l\rho_v\bar{u}_z^{\,2} - (\rho_l+\rho_v)\sigma\left(\alpha^2 - \frac{1}{R_0^{\,2}}\right)\right]^2 + [(\rho_l+\rho_v)\rho_l a]^2} + (\rho_l+\rho_v)\rho_l a}}{\sqrt{2}(\rho_l+\rho_v)}$$

Transient jet – Results1

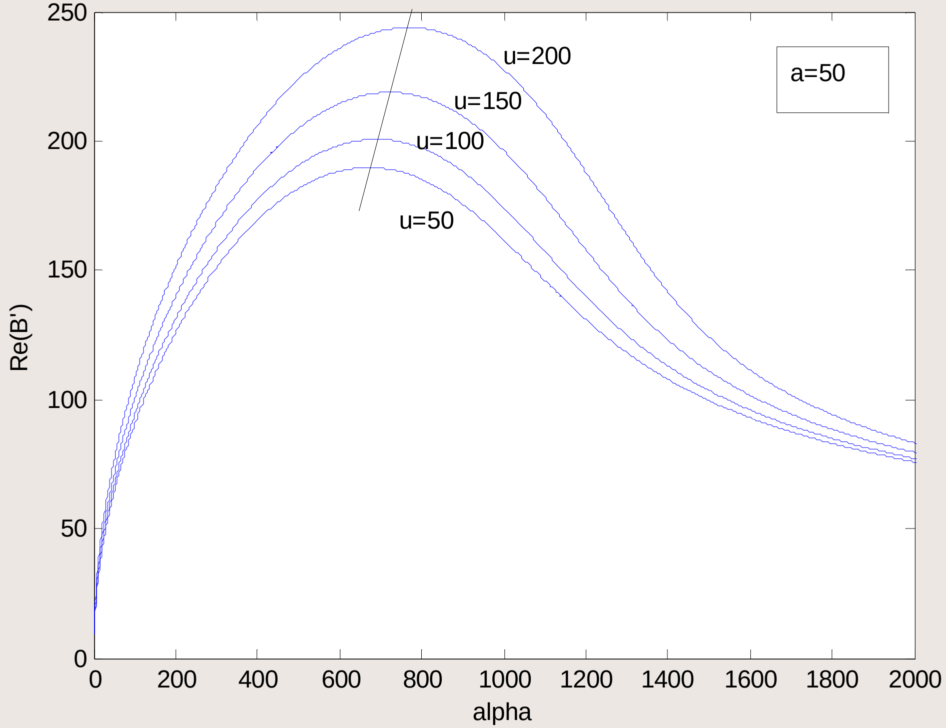
Effect of acceleration



Transient jet – Results2

Effect of velocity

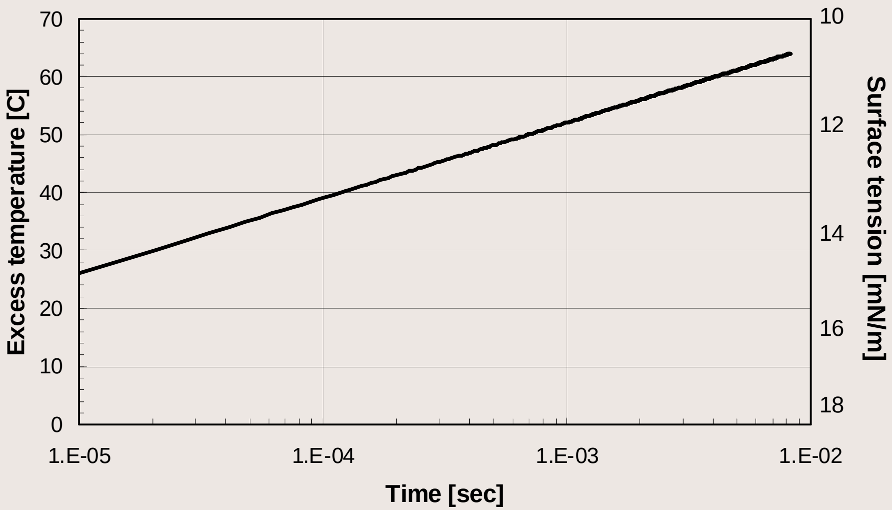
R0=0.001



Further application – Non-isothermal jets

Jet velocity ~ 10 - 100m/s
Jet diameter ~ 1mm
Ambient temperature excess ~ 1000K

Jet surface excess temperature



Coordinate system moves with the liquid phase: $\sigma(T) \rightarrow \sigma(t)$

$$(\rho_l + \rho_v)\left[\frac{d^2\beta(t)}{dt^2} + \left(\frac{d\beta(t)}{dt}\right)^2\right] + \rho_l \alpha\left(2i\bar{u}_z \frac{d\beta(t)}{dt} + i\frac{d\bar{u}_z}{dt} - \alpha\bar{u}_z^{\,2}\right) + \sigma\alpha\left(\alpha^2 - \frac{1}{R_0^{\,2}}\right) = 0$$

Closure

An analytic approach to analyse transient effects on jets instabilities:

- Transient jet velocity

- Transient surface tension (injection to hot ambient)