Global instability of flow over a rotating disc

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1 Local and global stability theory
Outline

1. Local and global stability theory
2. Rotating disc flow
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2. Rotating disc flow
3. A new global frequency selection mechanism
Stability of shear layers

- There are many shear layers that are either parallel, or approximately parallel, e.g.
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Stability of shear layers

- There are many shear layers that are either parallel, or approximately parallel, e.g.
  - channel flows,
  - boundary layers,

\[
\frac{\partial P}{\partial x} < 0 \quad \frac{\partial P}{\partial x} = 0 \quad \frac{\partial P}{\partial x} > 0
\]

\[
T_w > T_f
\]
unbounded shear layers,

jets       wakes       mixing layers
- unbounded shear layers,
  - jets
  - wakes
  - mixing layers

- Channel flows are exactly parallel.
unbounded shear layers,

Channel flows are exactly parallel.
The others may become ‘more parallel’ as the Reynolds number increases.
In ‘local’ theory, the basic flow is assumed parallel:

\[ \tilde{u} = U(y) + \epsilon u(y) \exp i(\alpha x - \omega t) \]
\[ \tilde{v} = \epsilon v(y) \exp i(\alpha x - \omega t) \]
\[ \tilde{p} = P(x) + \epsilon p(y) \exp i(\alpha x - \omega t) \]

giving an ODE for \( v \), and an eigenrelation between \( \alpha \) and \( \omega \).
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In ‘global’ theory, the basic flow is nonparallel:

\[ \tilde{u} = U(x, y) + \epsilon u(x, y) \exp(-i\omega_G t) \]
\[ \tilde{v} = V(x, y) + \epsilon v(x, y) \exp(-i\omega_G t) \]
\[ \tilde{p} = P(x, y) + \epsilon p(x, y) \exp(-i\omega_G t) \]

This gives a PDE for \( v \), and an eigenrelation for \( \omega_G \).
Local and global stability theory

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  giving a PDE for \( v \), and an eigenrelation for \( \omega_G \).

- How are local and global theories related when the basic flow varies slowly in the \( x \) direction?
Is $\omega_G$ the most unstable local frequency, i.e. $\omega$ when $\text{Im}(\omega)$ is maximized over $x$ and $\alpha$?
Global frequency selection criteria

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- No — this wave may propagate out of region of interest.
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convective instability

absolute instability
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  ![Diagram](image)

  - Let $\omega = \omega_0$, where $d\omega/d\alpha = x/t = 0$, be the local absolute frequency.
  - How do the rays curve when the flow varies with $x$?
Sketch of rotating-disc flow
A rotating-disc experiment
## Laminar-turbulent transition on the rotating disc

<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>$Re_t$</th>
<th>Method</th>
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<tr>
<td>Gregory, Stuart &amp; Walker</td>
<td>1955</td>
<td>530</td>
<td>Visual, China-clay</td>
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<tr>
<td>Federov <em>et al.</em></td>
<td>1976</td>
<td>515</td>
<td>Visual, napthalene</td>
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<tr>
<td>Clarkson, Chin &amp; Shacter</td>
<td>1980</td>
<td>562</td>
<td>Visual, dye</td>
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<tr>
<td>Cobb &amp; Saunders</td>
<td>1956</td>
<td>490</td>
<td>Heat transfer</td>
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<tr>
<td>Chin &amp; Litt</td>
<td>1972</td>
<td>510</td>
<td>Mass transfer</td>
</tr>
<tr>
<td>Gregory &amp; Walker</td>
<td>1960</td>
<td>505</td>
<td>Pressure probe</td>
</tr>
<tr>
<td>Smith</td>
<td>1946</td>
<td>557</td>
<td>Hot-wire</td>
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<tr>
<td>Kobayashi <em>et al.</em></td>
<td>1980</td>
<td>500</td>
<td>Hot-wire</td>
</tr>
<tr>
<td>Malik, Wilkinson &amp; Orszag</td>
<td>1981</td>
<td>520</td>
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<tr>
<td>Wilkinson &amp; Malik</td>
<td>1985</td>
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<td>Lingwood</td>
<td>1996</td>
<td>508</td>
<td>Hot-wire</td>
</tr>
<tr>
<td>Othman &amp; Corke</td>
<td>2006</td>
<td>539</td>
<td>Hot-wire</td>
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</table>
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This implies that rotating disc flow is **globally stable**.

We introduce a new global frequency selection mechanism.
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We introduce a new global frequency selection mechanism.
It is relevant to rotating disc flow.
It predicts global instability.
It may explain the observed variation in $Re_t$ in experiments.
In some circumstances, the envelope, $A$, of a wavepacket

$$A(X, T) \exp i(\alpha x - \omega t)$$

satisfies the linearized complex Ginzburg-Landau eqn:

$$\frac{\partial A}{\partial T} + U \frac{\partial A}{\partial X} = \mu A + \gamma \frac{\partial^2 A}{\partial X^2}. $$
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An amplitude equation approach

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- Re($\gamma$) > 0 is the rate of spreading (dispersion/diffusion).
A model flow

- Taking the coefficients $U = U(X)$, $\mu = \mu(X)$ and $\gamma = \gamma(X)$ models the propagation of a wavepacket through a spatially varying flow.
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where $\epsilon \ll 1$ implies a slowly varying flow.

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- Hunt & Crighton (1991) give an exact impulsive solution for these coefficients.
Local results

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- Local instability is independent of $\delta$. 

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- As $T \to \infty$ for any fixed finite $X$,

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- Global instability for \( \delta^2 < 1 \).
- Global decay for \( \delta^2 > 1 \).
An example

For $\epsilon = 0.01$, there is local convective instability for $X > 0$, and local absolute instability for $X > 25$. 
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- Contours of $|A|$:

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- Contours of $|A|$:  
  \begin{align*}
  \delta = 0: \text{global instability} \\
  \delta = 2: \text{global decay}
  \end{align*}
- Davies, Thomas & Carpenter (2007) argued that rotating disc flow has strong enough detuning (large enough $\delta$) to make it globally stable, despite existence of local absolute instability.
Creation of global instability

- Remember that discs in experiments have finite radius!
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Keep $A \to 0$ as $X \to -\infty$, but let $A = 0$ at $X = h$ (e.g. at edge of disc).
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- Keep \( A \to 0 \) as \( X \to -\infty \), but let \( A = 0 \) at \( X = h \) (e.g. at edge of disc).
- This creates a discrete spectrum of global modes
  \[
  A = \psi(X) \exp(-i\omega_G T) \text{ where }
  \psi'' - \psi' + [i\omega_G + (1 + i\delta)\epsilon X]\psi = 0.
  \]
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The eigenvalues are
\[
\omega_G = i \left[ (1 + i\delta)\epsilon h - 1/4 + \epsilon^{2/3}(1 + i\delta)^{2/3}b_n \right] = \omega_0(h) + O(\epsilon^{2/3})
\]

where \( \text{Ai}(b_n) = 0 \), i.e. \( b_1 \approx -2.34, \ b_2 \approx -4.09 \), etc.
Creation of global instability

- Remember that discs in experiments have finite radius!
- Hunt & Crighton’s solution has b.c.s $A \to 0$ as $X \to \pm \infty$.
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- This creates a discrete spectrum of global modes $A = \psi(X) \exp(-i\omega_G T)$ where

$$\psi'' - \psi' + [i\omega_G + (1 + i\delta)\epsilon X]\psi = 0.$$ 

- The eigenvalues are

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where $\text{Ai}(b_n) = 0$, i.e. $b_1 \approx -2.34$, $b_2 \approx -4.09$, etc.
- Local absolute instability at disc edge $\Rightarrow$ global instability.
An example

Let $\epsilon = 0.01$, $\delta = 2$ and $A \to 0$ as $X \to -\infty$.

Global decay when $A \to 0$ as $X \to \infty$.

Global instability when $A = 0$ at $X = 100$. 
Stabilizing nonlinearity

Consider

\[
\frac{\partial A}{\partial T} + \frac{\partial A}{\partial X} = 0.01(1 + i\delta)XA + \frac{\partial^2 A}{\partial X^2} - |A|^2 A.
\]
Qualitative behaviour of front

\[ X = h \]

There is no front for \( h < h_c \), where \( h_c > X_{C/A} \).
Qualitative behaviour of front

There is no front for $h < h_c$, where $h_c > X_{C/A}$.

As $h$ passes through $h_c$ the front appears and moves inwards, approaching the convective-absolute transition location for large $h$. 

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Rotating disc transition experiments

$Re_{\text{trans}}$

$Re_{\text{edge}}$

Disc

Convective instability

Global instability

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Conclusions

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- The global frequency can then be driven by the out-flow local absolute instability.
- This provides a possible mechanism for global instability in rotating disc flow.
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- $Re_t$ for the disc is correlated to $Re_{\text{edge}}$. 
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- $Re_t$ for the disc is correlated to $Re_{\text{edge}}$.
- $Re_t$ follows the same qualitative dependence as the front in the nonlinear global mode theory.