Global instability of flow over a rotating disc

Jonathan Healey

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Outline

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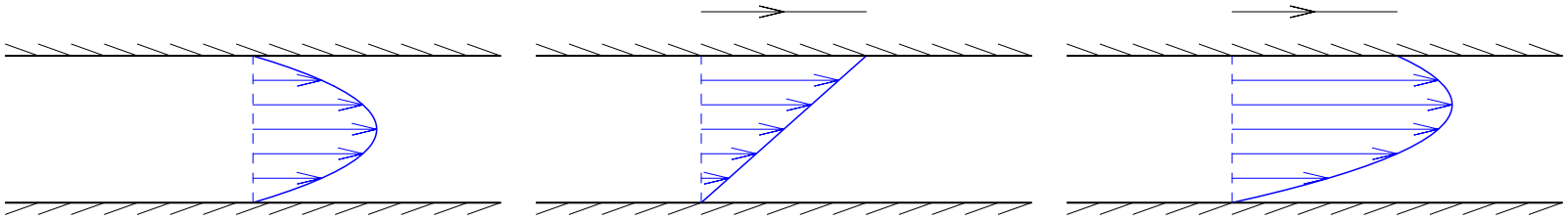

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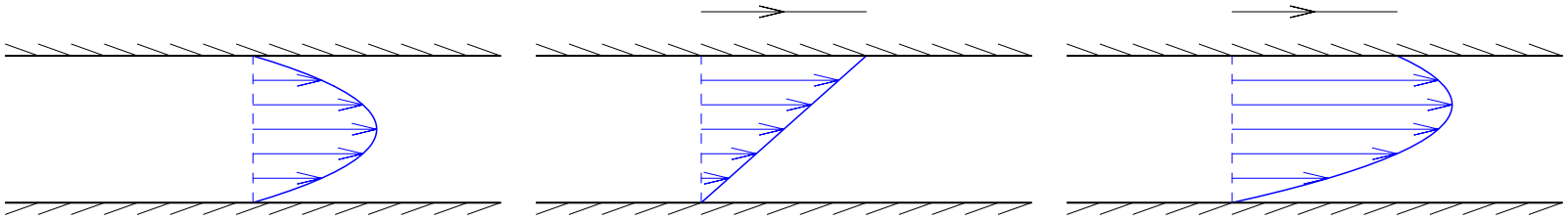

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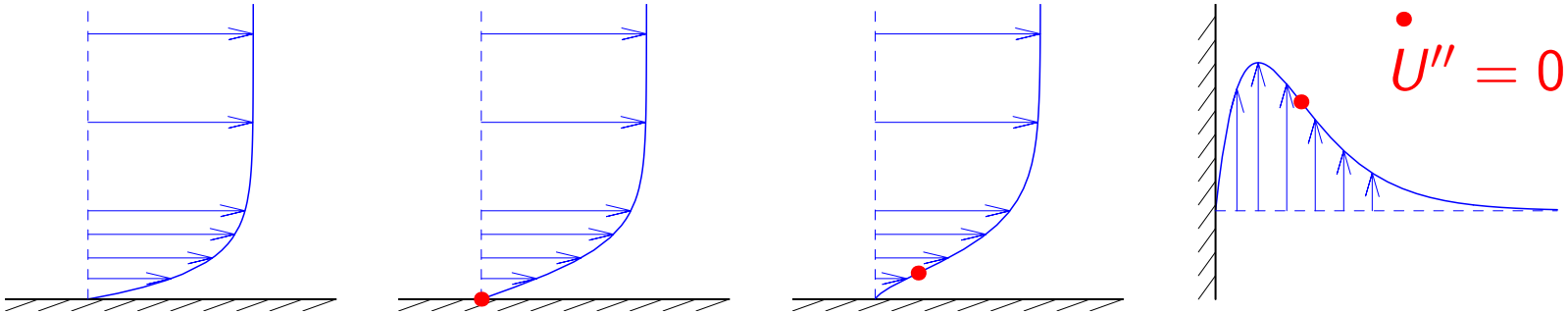

$\partial P/\partial x < 0$ $\quad$ $\partial P/\partial x = 0$ $\quad$ $\partial P/\partial x > 0$ $\quad$ $T_w > T_f$

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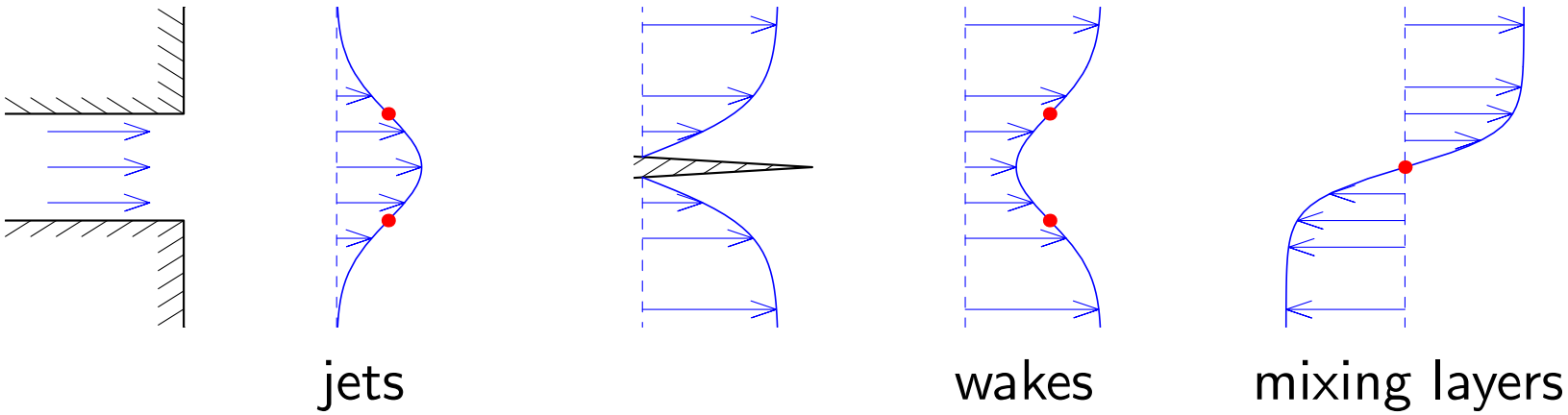

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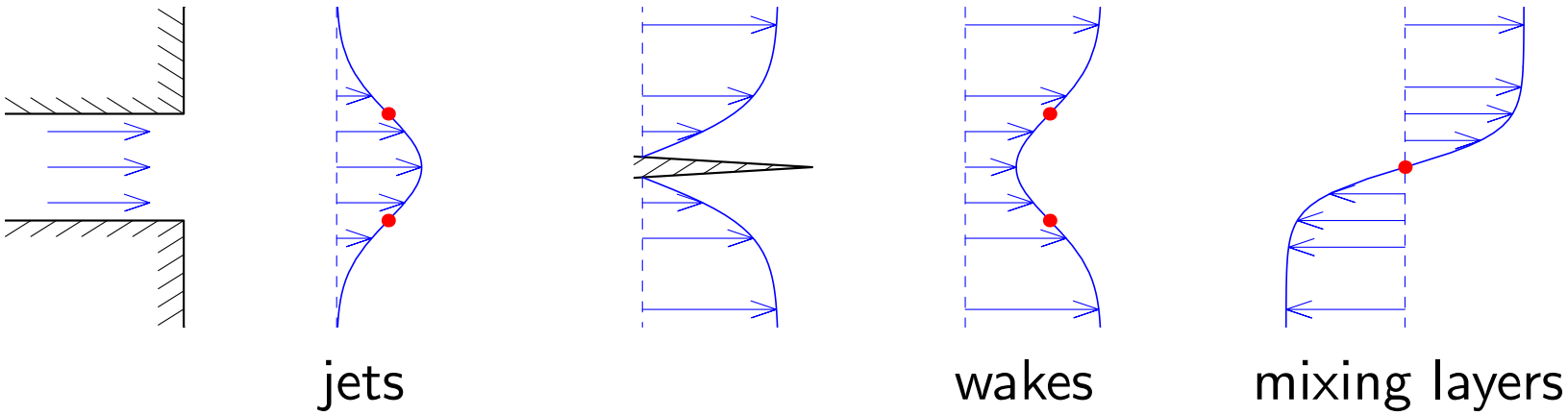

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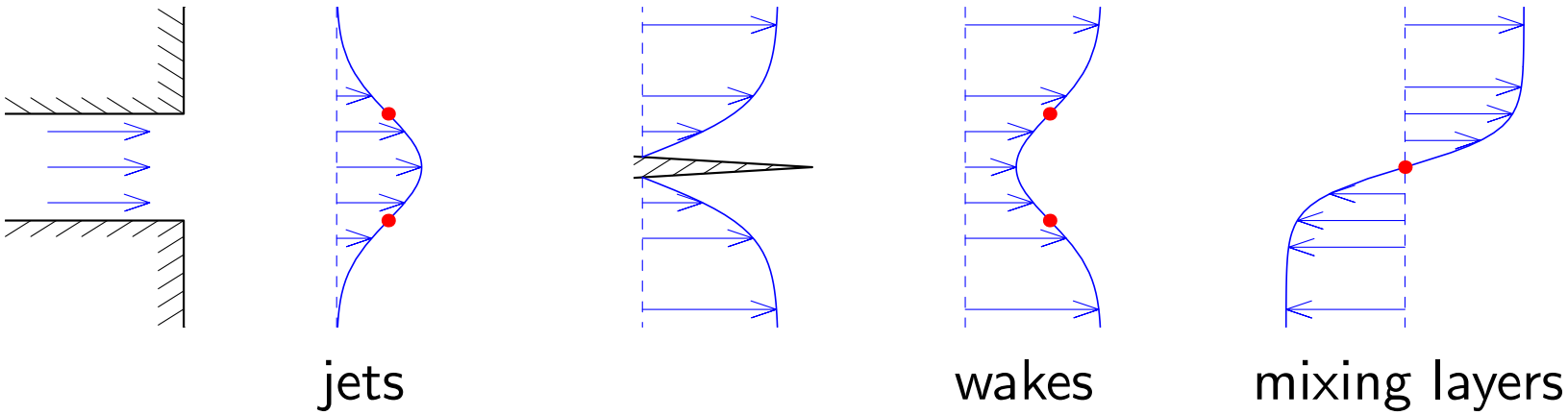

- Channel flows are exactly parallel.
- The others may become 'more parallel' as the Reynolds number increases.

Local and global stability theory

- In 'local' theory, the basic flow is assumed parallel:

$$\begin{aligned}\tilde{u} &= U(y) + \epsilon u(y) \exp \mathrm{i}(\alpha x - \omega t) \\ \tilde{v} &= \epsilon v(y) \exp \mathrm{i}(\alpha x - \omega t) \\ \tilde{p} &= P(x) + \epsilon p(y) \exp \mathrm{i}(\alpha x - \omega t)\end{aligned}$$

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- In 'global' theory, the basic flow is nonparallel:

$$\begin{aligned}\tilde{u} &= U(x,y) + \epsilon u(x,y) \exp(-\mathrm{i}\omega_G t) \\ \tilde{v} &= V(x,y) + \epsilon v(x,y) \exp(-\mathrm{i}\omega_G t) \\ \tilde{p} &= P(x,y) + \epsilon p(x,y) \exp(-\mathrm{i}\omega_G t)\end{aligned}$$

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- How are local and global theories related when the basic flow varies slowly in the x direction?

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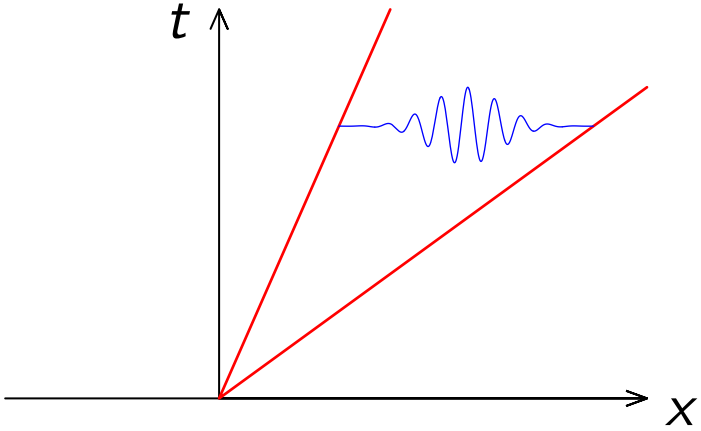
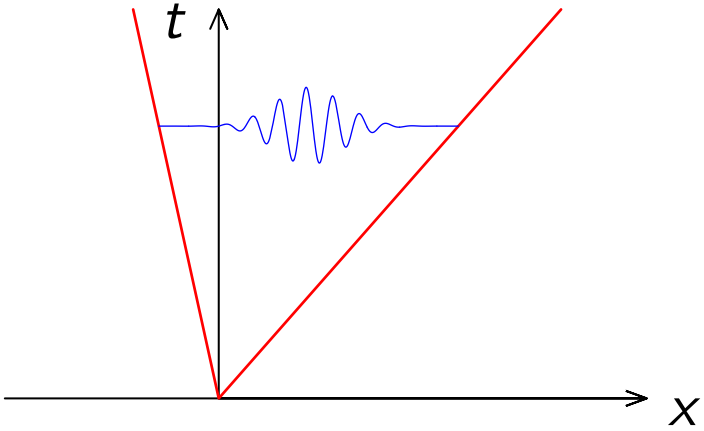


convective instability



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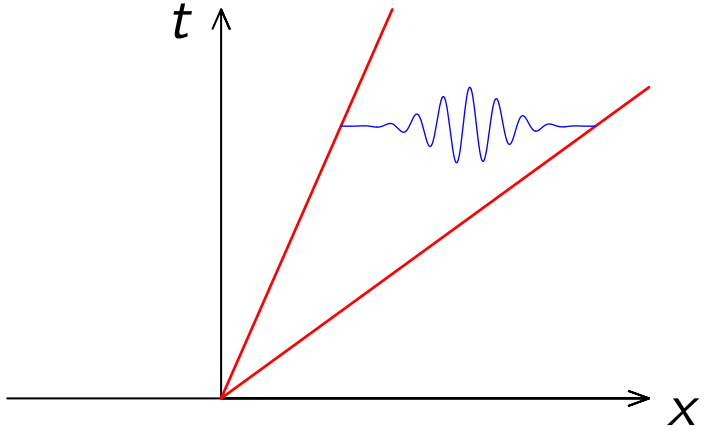

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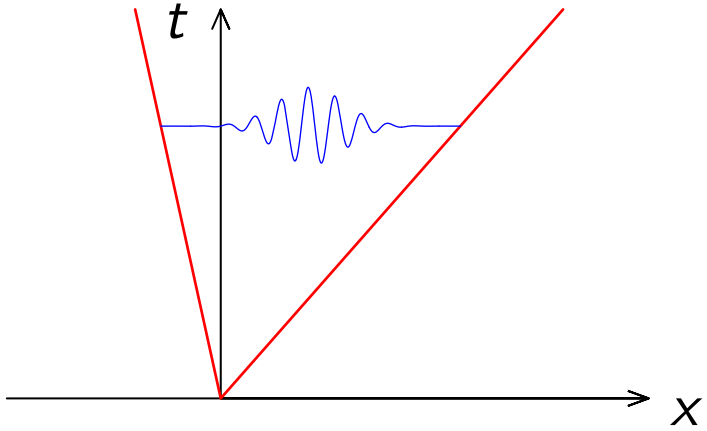

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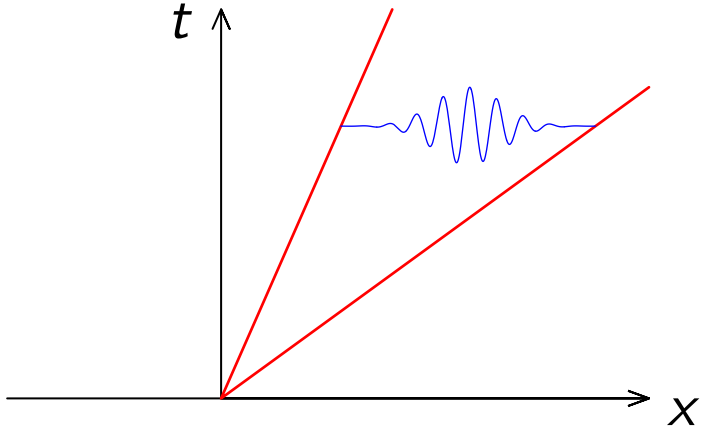

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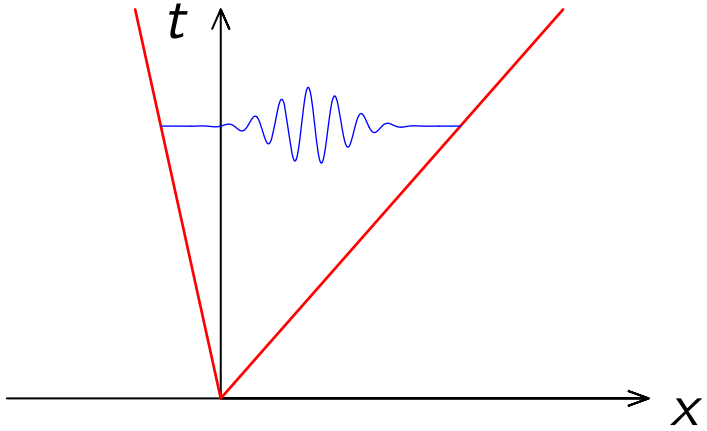

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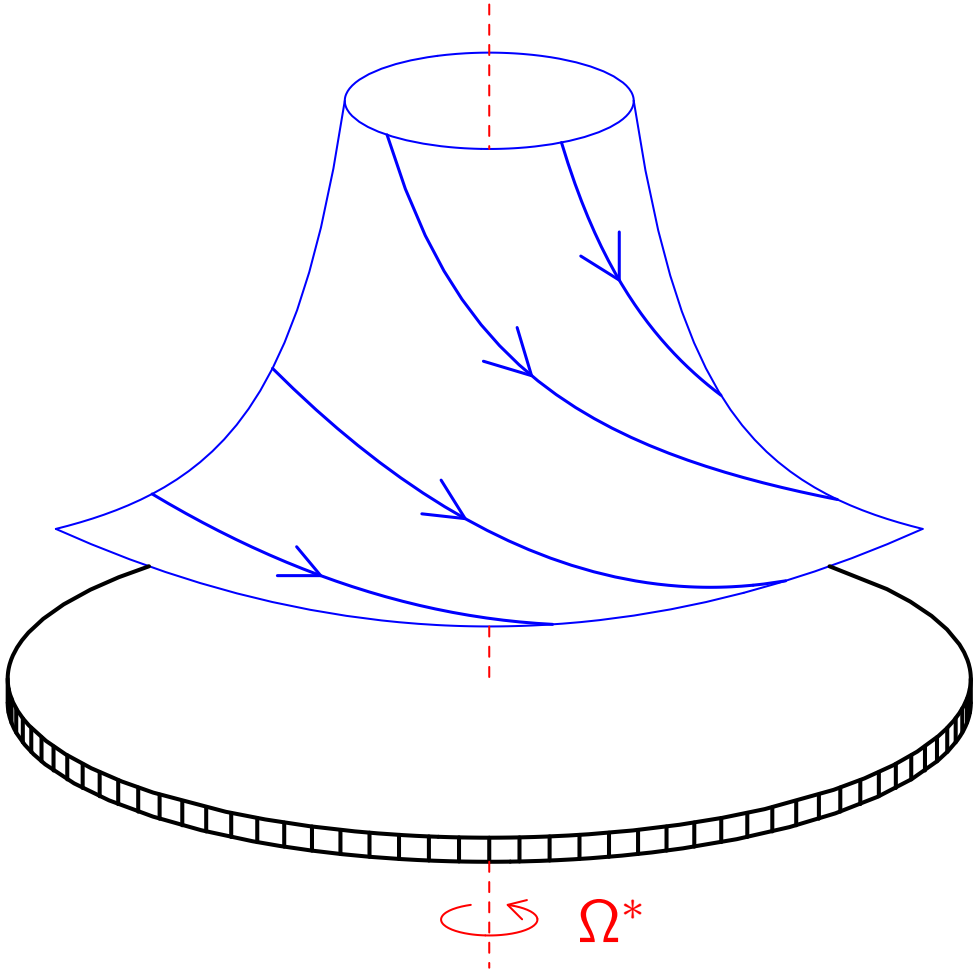
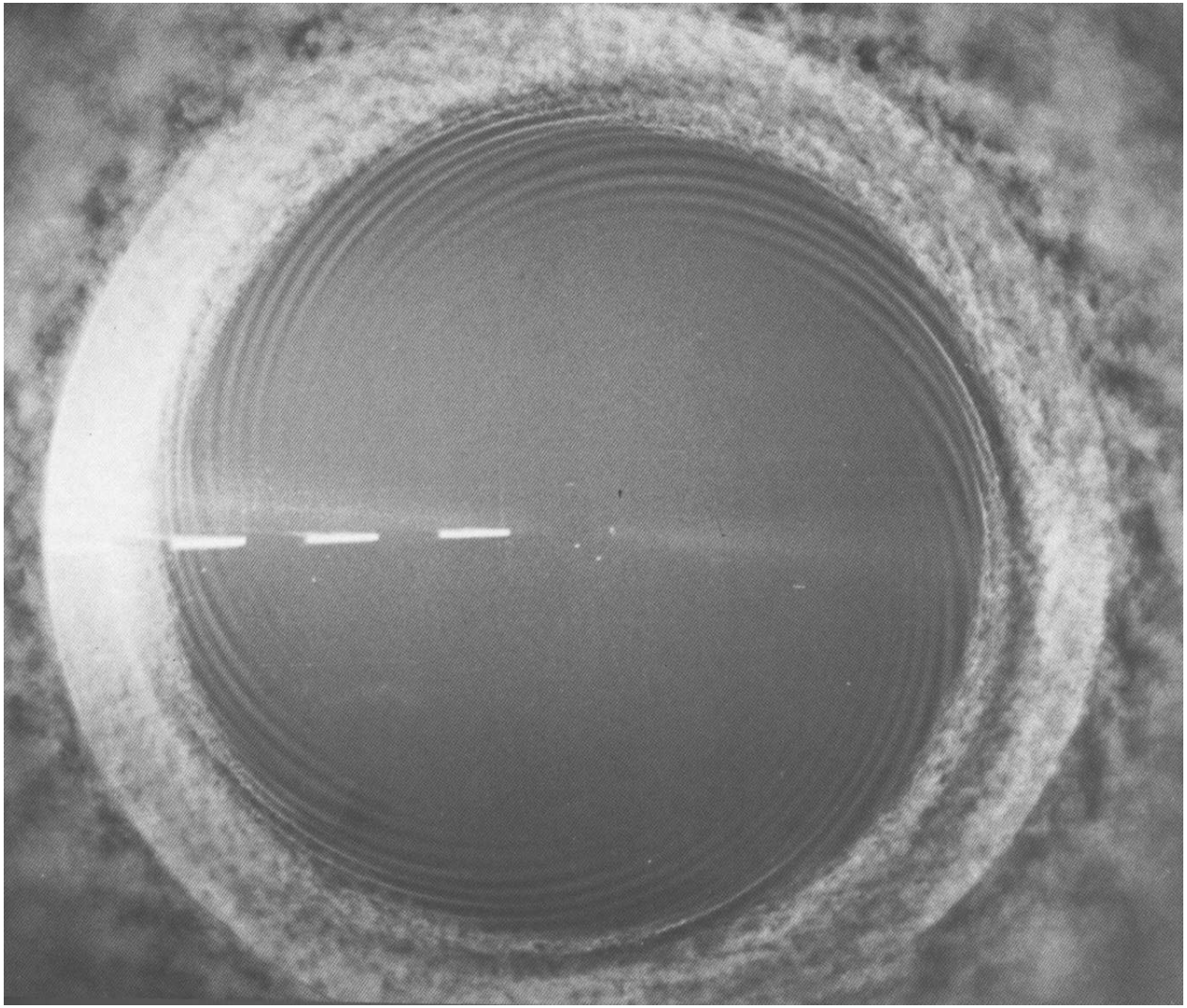
- How do the rays curve when the flow varies with x?

Sketch of rotating-disc flow



A rotating-disc experiment



Laminar-turbulent transition on the rotating disc

	year	Re_t	
Gregory, Stuart & Walker	(1955)	530	Visual, China-clay
Federov *et al.*	(1976)	515	Visual, napthalene
Clarkson, Chin & Shacter	(1980)	562	Visual, dye
Cobb & Saunders	(1956)	490	Heat transfer
Chin & Litt	(1972)	510	Mass transfer
Gregory & Walker	(1960)	505	Pressure probe
Smith	(1946)	557	Hot-wire
Kobayashi *et al.*	(1980)	500	Hot-wire
Malik, Wilkinson & Orszag	(1981)	520	Hot-wire
Wilkinson & Malik	(1985)	550	Hot-wire
Lingwood	(1996)	508	Hot-wire
Othman & Corke	(2006)	539	Hot-wire

Transition related to onset of absolute instability?

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- Othman & Corke (2006) observed, in an experiment, wavepackets propagating through a laminar flow with $Re > 507$, confirming global stability.

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- It may explain the observed variation in Re_t in experiments.

An amplitude equation approach

- In some circumstances, the envelope, A, of a wavepacket

$$A(X,T)\exp \mathrm{i}(\alpha x - \omega t)$$

satisfies the linearized complex Ginzburg-Landau eqn:

$$\frac{\partial A}{\partial T} + U\frac{\partial A}{\partial X} = \mu A + \gamma \frac{\partial^2 A}{\partial X^2}.$$

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- $\mathrm{Re}(\gamma) > 0$ is the rate of spreading (dispersion/diffusion).

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- Local instability is independent of δ.

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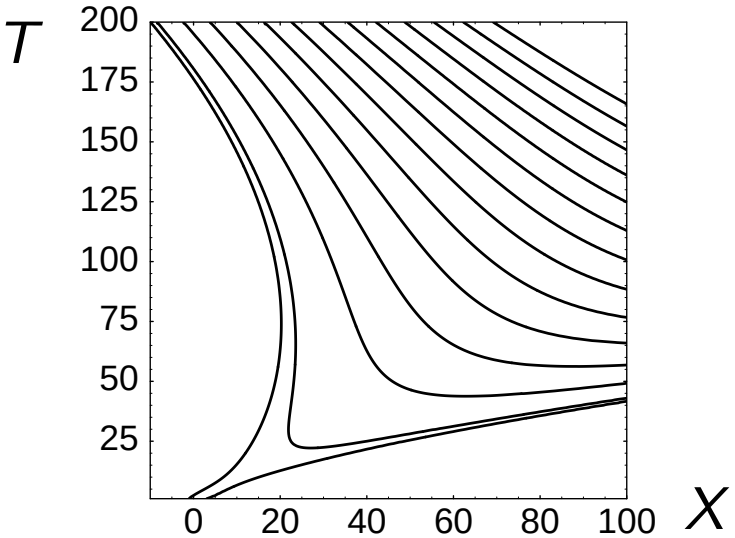
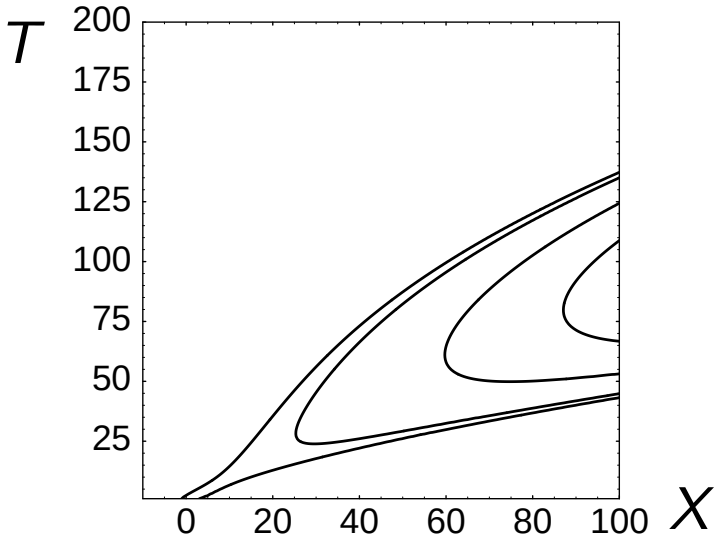


$\delta = 0$: global instability



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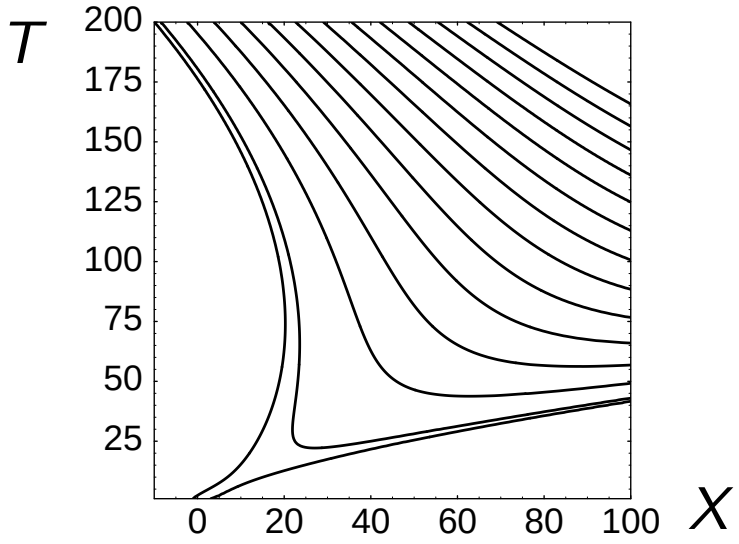

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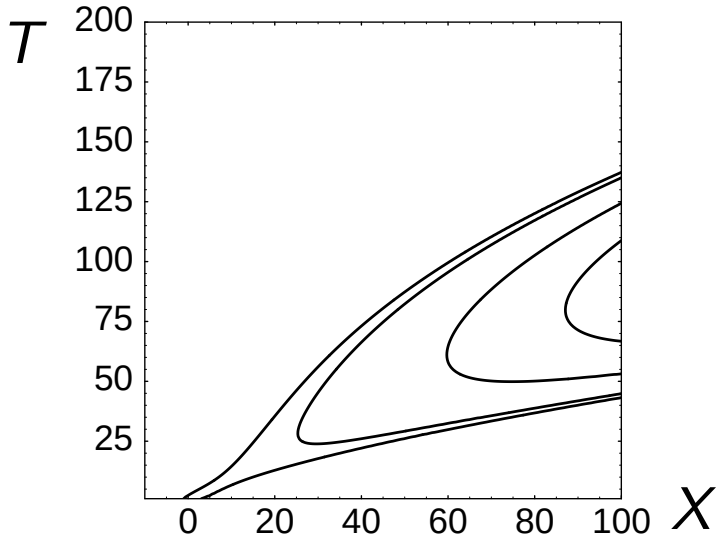

$\delta = 2$: global decay

- Davies, Thomas & Carpenter (2007) argued that rotating disc flow has strong enough detuning (large enough δ) to make it globally stable, despite existence of local absolute instability.

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- This creates a discrete spectrum of global modes $A = \psi(X)\exp(-\mathrm{i}\omega_G T)$ where

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- The eigenvalues are

$$\omega_G = \mathrm{i}\left[(1+\mathrm{i}\delta)\epsilon h - 1/4 + \epsilon^{2/3}(1+\mathrm{i}\delta)^{2/3} b_n\right] = \omega_0(h) + O(\epsilon^{2/3})$$

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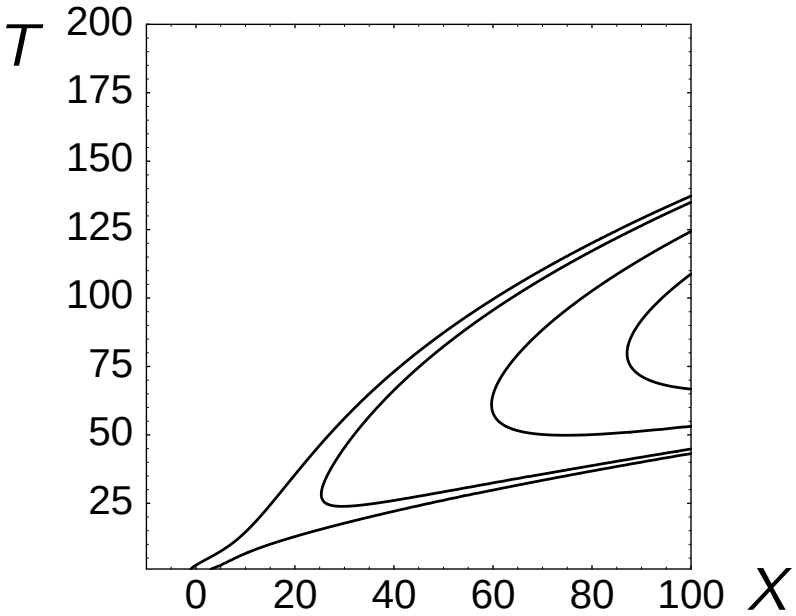
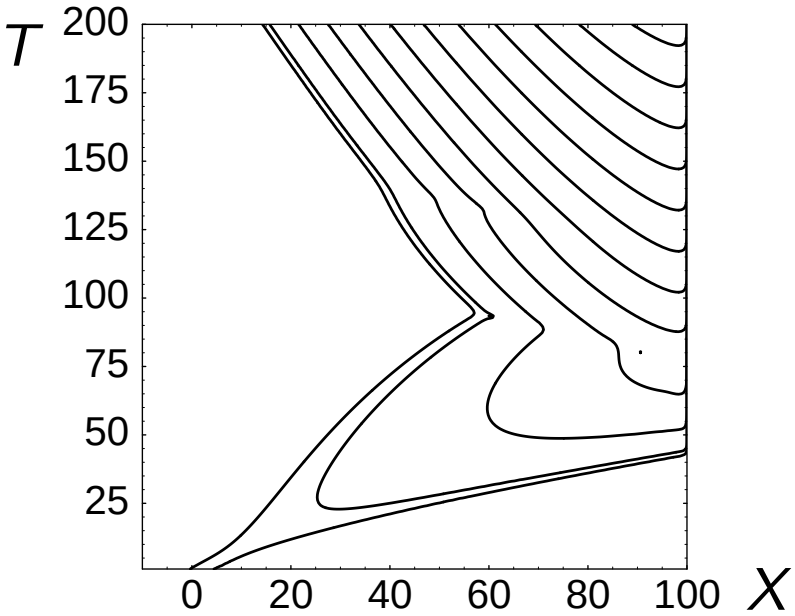
- Local absolute instability at disc edge $\Rightarrow$ global instability.

An example

Let $\epsilon = 0.01$, $\delta = 2$ and $A \to 0$ as $X \to -\infty$.



Global decay when $A \to 0$ as $X \to \infty$.

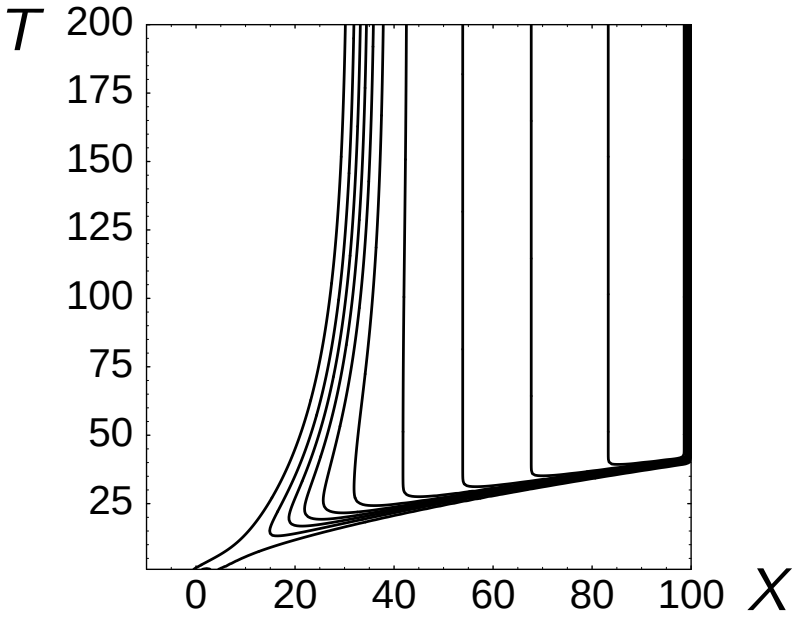


Global instability when $A = 0$ at $X = 100$.

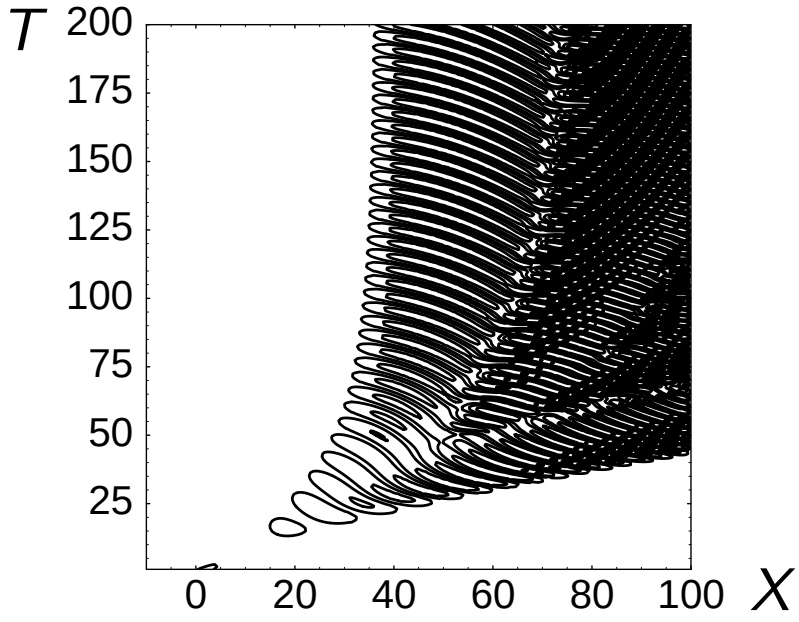
Stabilizing nonlinearity

Consider

$$\frac{\partial A}{\partial T} + \frac{\partial A}{\partial X} = 0.01(1 + \mathrm{i}\delta)XA + \frac{\partial^2 A}{\partial X^2} - |A|^2 A.$$

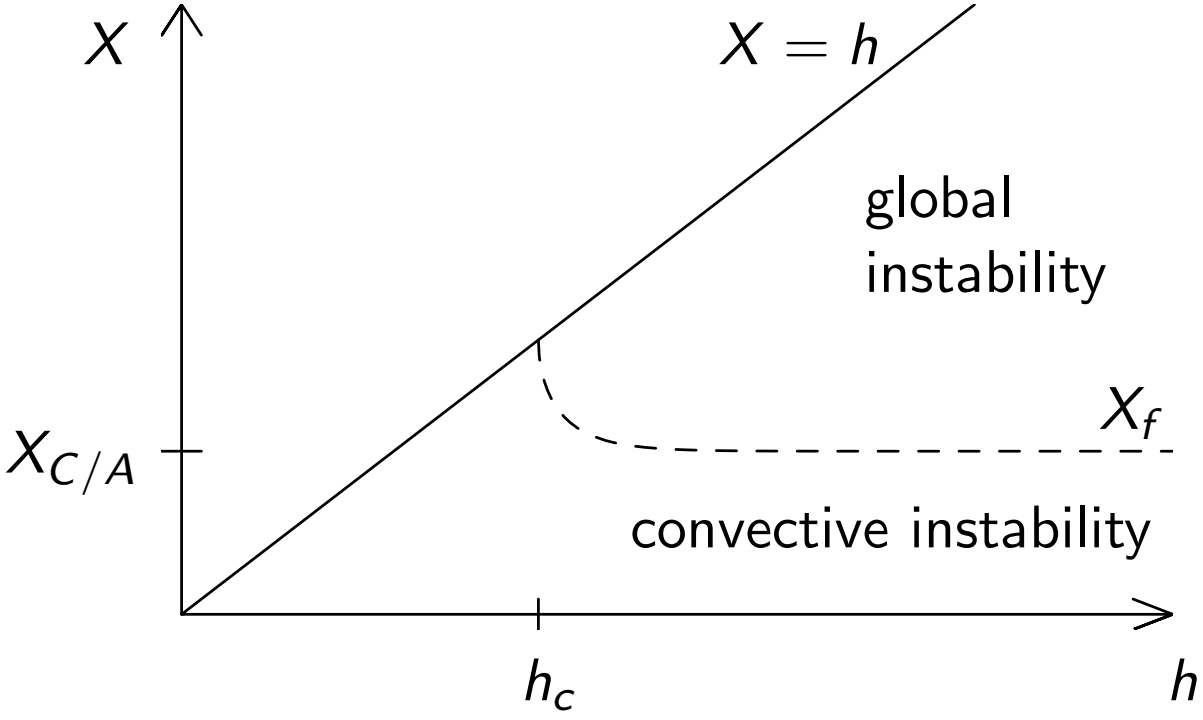


$\delta = 0$



$\delta = 2$

Qualitative behaviour of front



- There is no front for $h < h_c$, where $h_c > X_{C/A}$.

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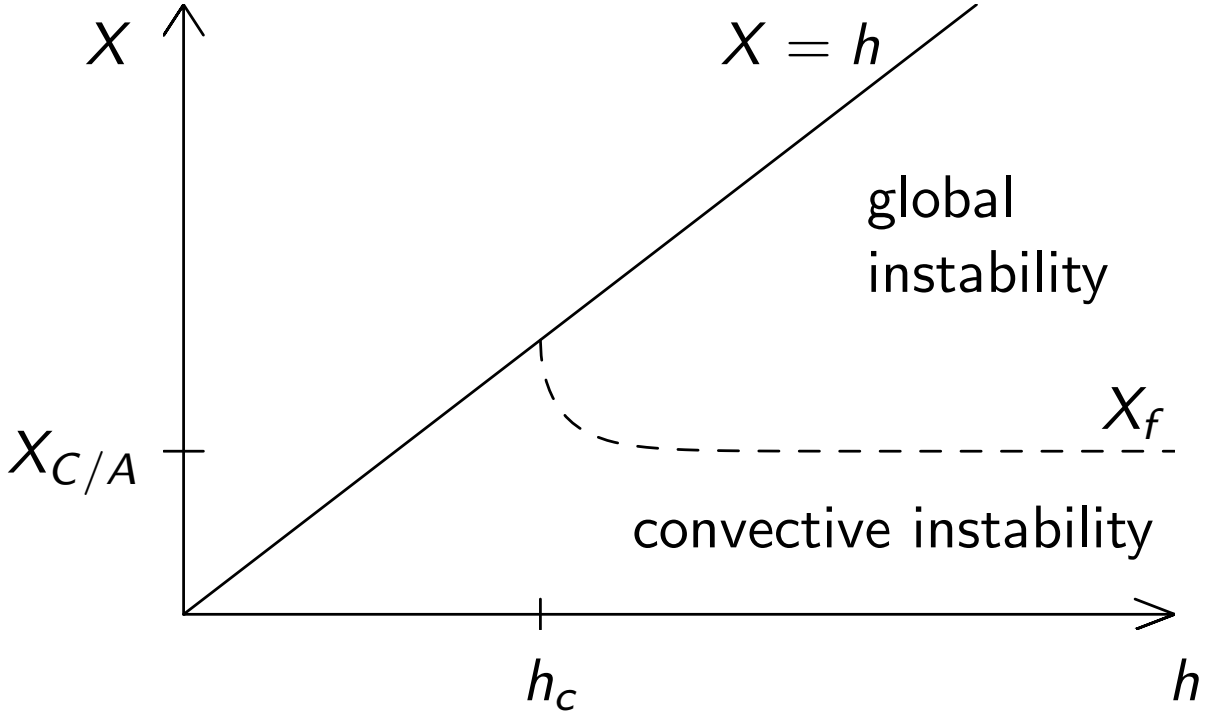
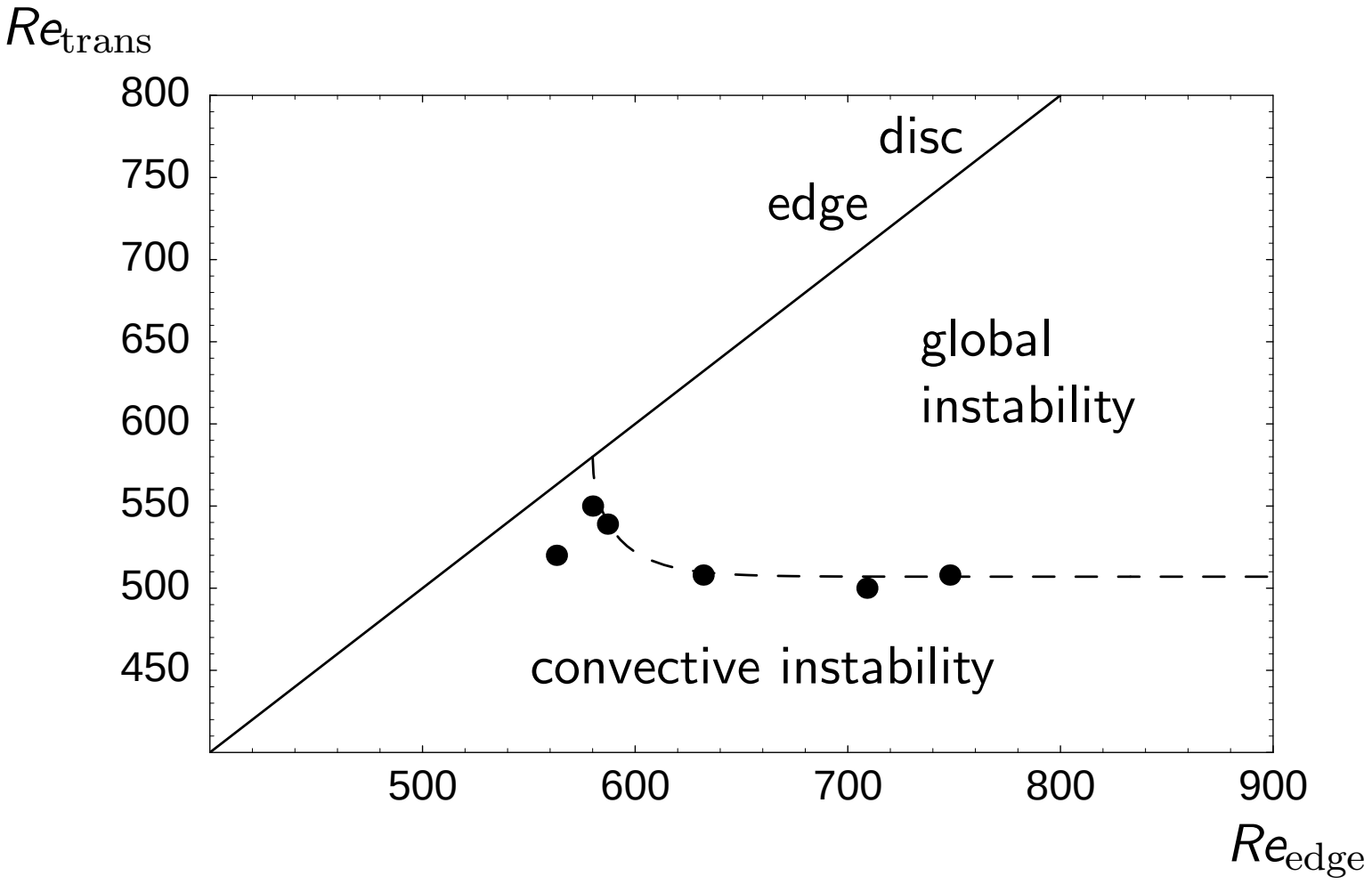


- There is no front for $h < h_c$, where $h_c > X_{C/A}$.
- As h passes through h_c the front appears and moves inwards, approaching the convective-absolute transition location for large h.

Rotating disc transition experiments



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- The global frequency can then be driven by the out-flow local absolute instability.
- This provides a possible mechanism for global instability in rotating disc flow.
- Re_t for the disc is correlated to Re_{edge}.
- Re_t follows the same qualitative dependence as the front in the nonlinear global mode theory.