Destabilizing effect of confinement on mixing layers

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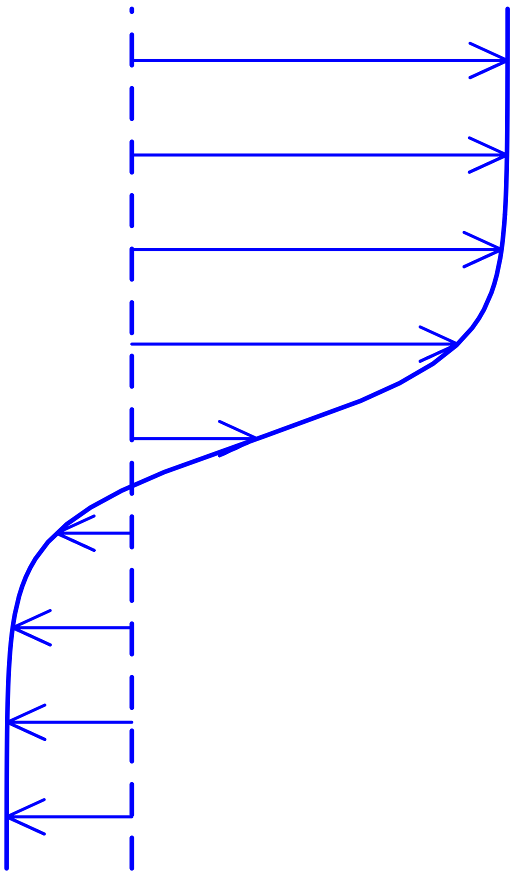
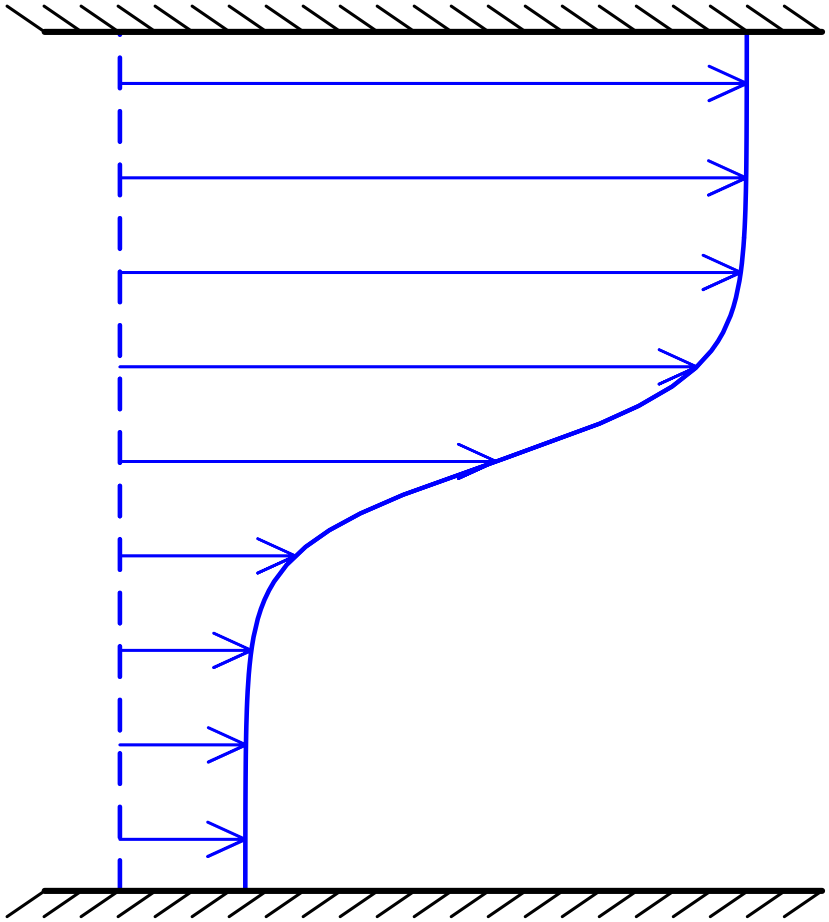
Mixing layers



Unconfined counter-flow
mixing layer



Confined co-flow
mixing layer

Basic flow

- Let $U_1^* > 0$ be the uniform flow in the upper layer.
- Let U_2^* be the uniform flow in the lower layer.
- $U_2^* < 0 \Rightarrow$ counter-flow; $U_2^* > 0 \Rightarrow$ co-flow.
- Nondimensionalize using the mean $(U_1^* + U_2^*)/2$, then the basic flow is

$$U(y) = 1 + r \tanh(y/2)$$

where

$$r = \frac{U_1^* - U_2^*}{U_1^* + U_2^*}.$$

- $0 < r < 1 \Rightarrow$ co-flow; $r > 1 \Rightarrow$ counter-flow.

Linearized waves

- Add a small disturbance to a parallel shear layer:

$$\begin{aligned}\tilde{u} &= U(y) + \epsilon u(y) \exp \mathrm{i}(\alpha x - \omega t)\\ \tilde{v} &= \epsilon v(y) \exp \mathrm{i}(\alpha x - \omega t)\\ \tilde{p} &= \epsilon p(y) \exp \mathrm{i}(\alpha x - \omega t)\end{aligned}$$

  where $\epsilon \ll 1$.
- Substitute into the Navier–Stokes equations.
- Neglect $O(\epsilon^2)$ terms (linearize).
- Eliminate u and p to give, in inviscid limit, the Rayleigh equation:

$$(U - c)(v'' - \alpha^2 v) - U'' v = 0$$

  where $c = \omega/\alpha$, $v = 0$ on boundaries.

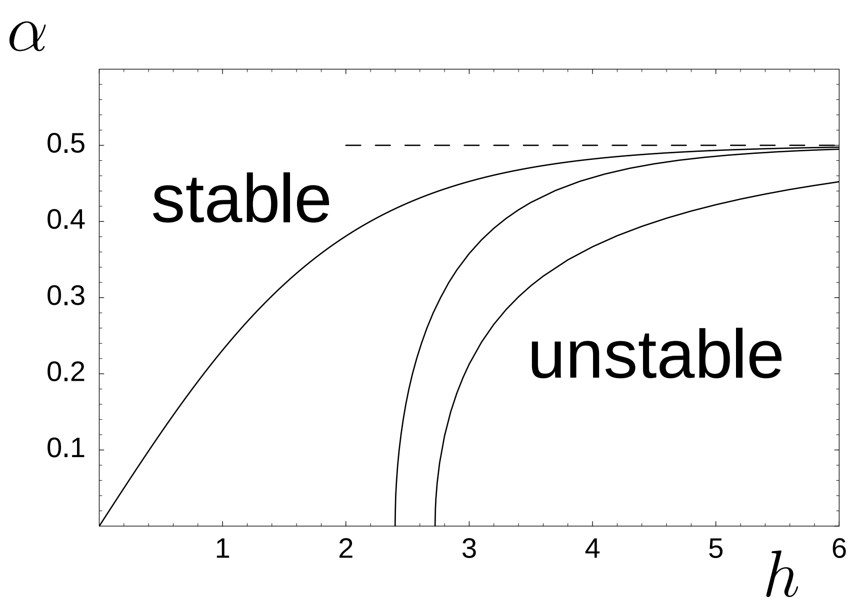
Temporal instability

- A spatially localized initial condition can be expressed as a superposition of normal modes with real α.
- Each normal mode evolves with an ω satisfying the dispersion relation.
- If there exists a real α with $\omega_i > 0$, then there is growth in time:

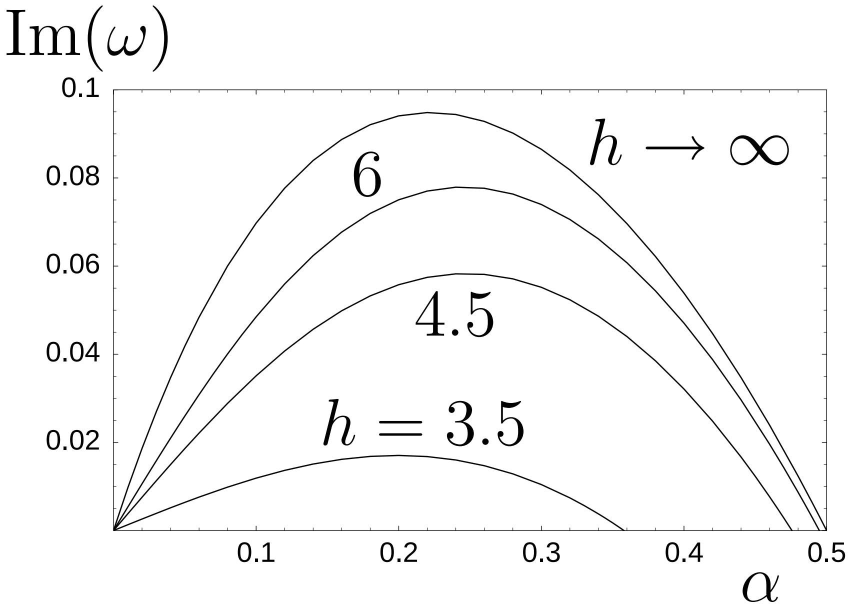
$$\exp(-\mathrm{i}\omega t) = (\exp -\mathrm{i}\omega_r t)(\exp \omega_i t).$$

But isn't confinement stabilizing?

- In the confined problem, plates are placed at $y = \pm h$ (asymmetric confinement will be considered later).
- Boundary conditions for Rayleigh equation become $v(\pm h) = 0$.



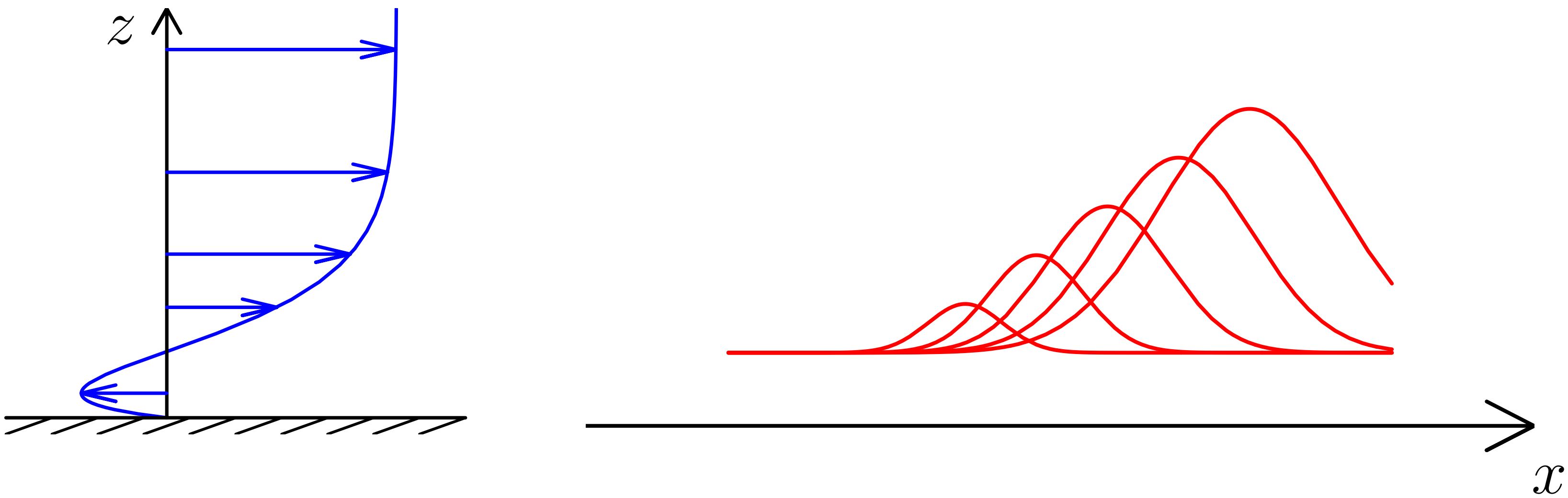
neutral curves



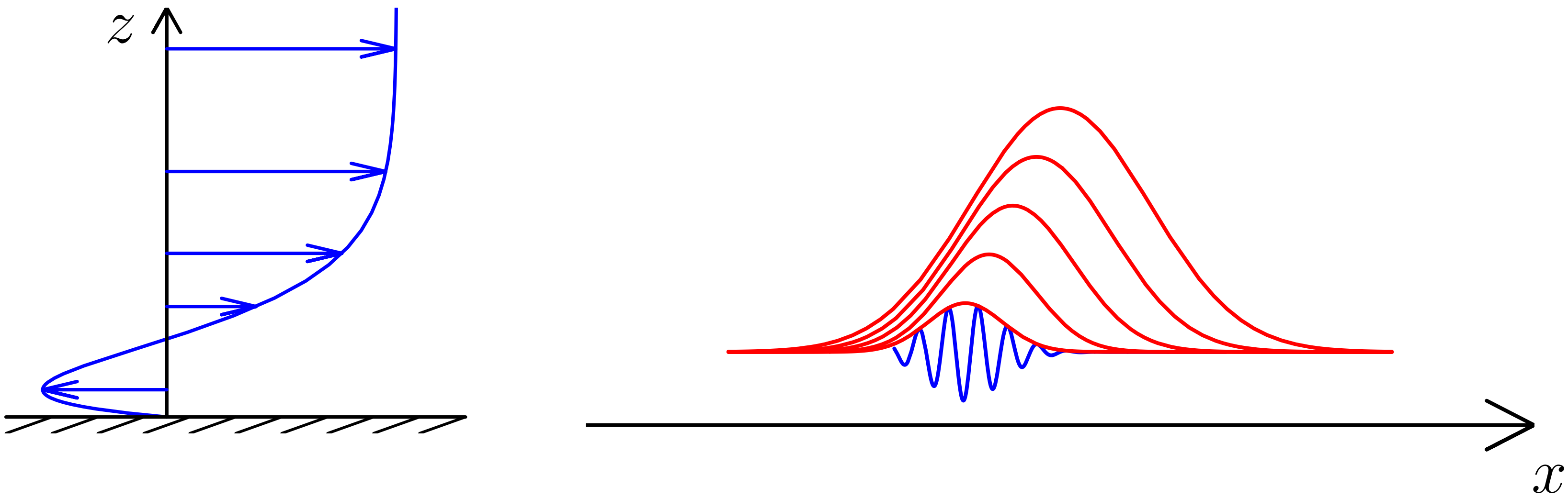
growth rates

Absolute and convective instabilities

Convective instability:

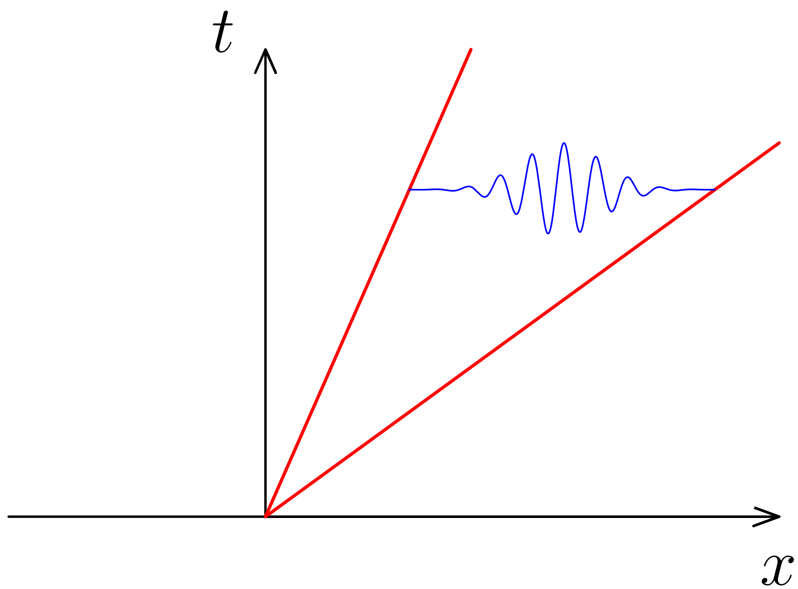


Absolute instability:



Space-time diagrams

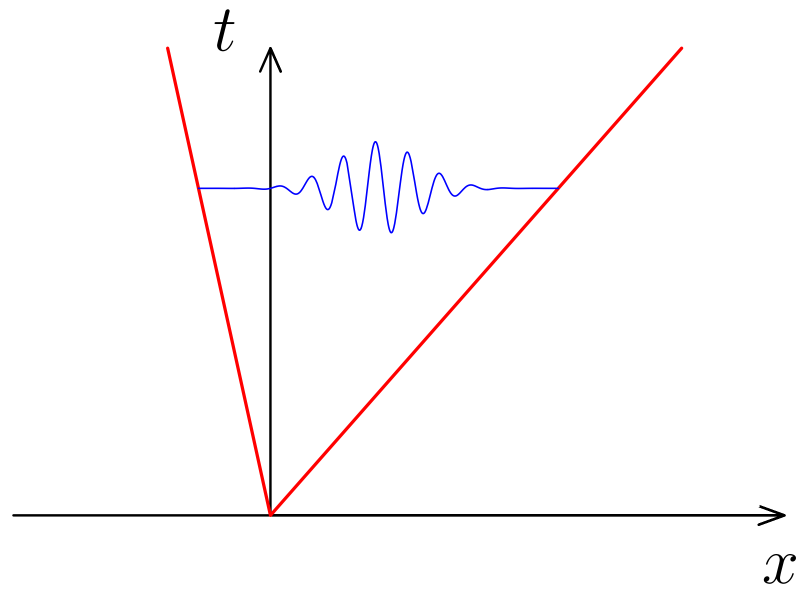
Convective instability



Disturbance grows as it propagates away, eventually leaving flow undisturbed.

Flow acts as spatial amplifier of transients.

Absolute instability



Disturbance grows in time everywhere.

Flow can act as self-excited oscillator.

Impulse calculations

- A wavepacket is constructed from a superposition of normal modes $v(y)\exp\mathrm{i}[\alpha x - \omega(\alpha)t]$ of the form:

$$\hat{v}(x,y,t) = \int_A v(y)\exp\phi t\,\mathrm{d}\alpha$$

where

$$\phi(\alpha) = \mathrm{i}\left[\alpha\frac{x}{t} - \omega(\alpha)\right].$$

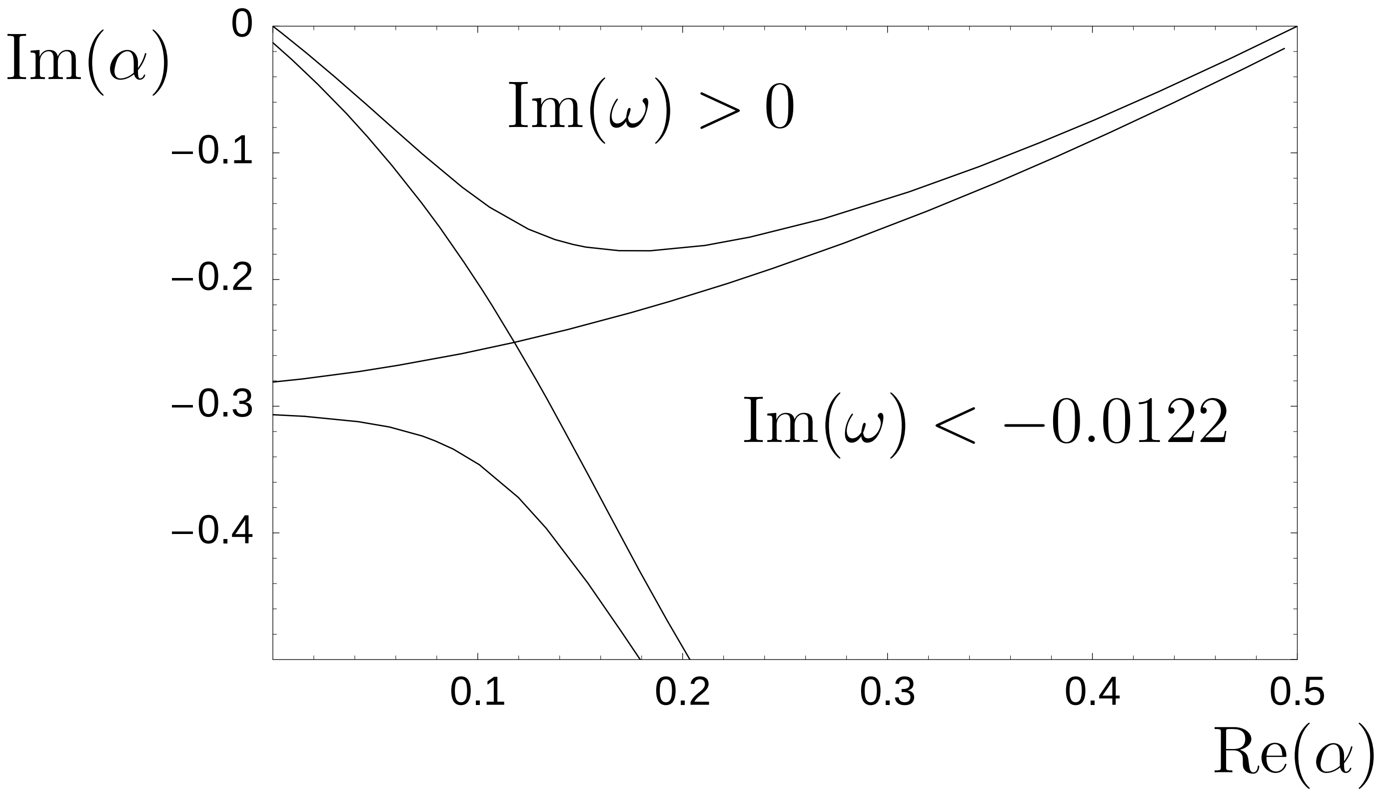
- In the limit $t \to \infty$ this integral is dominated by the contribution from a saddle-point, at which

$$\frac{\mathrm{d}\phi}{\mathrm{d}\alpha} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\omega}{\mathrm{d}\alpha} = \frac{x}{t}.$$

- There is absolute instability if $\mathrm{Im}(\omega) > 0$ at the dominant saddle (pinch-point) for $x/t = 0$.

Saddle point for an unconfined flow

$r = 1.25,\ f(y) = \tanh(y/2)$



- Huerre & Monkewitz (1985) found absolute instability for $r > 1.315$.
- They also noted that for $r < 0.84$ the saddle point enters the left half plane.

Branch-cuts on imaginary axis

- Outside a shear layer, $U \to \mathrm{const.}$, $U'' \to 0$, and Rayleigh equation reduces to

$$v'' - \alpha^2 v = 0$$

with solution

$$v(y) = C_1 \exp(-\alpha y) + C_2 \exp(\alpha y).$$

- Or, more precisely,

$$v(y) = C_1 \exp\left(-\sqrt{\alpha^2} y\right) + C_2 \exp\left(\sqrt{\alpha^2} y\right)$$

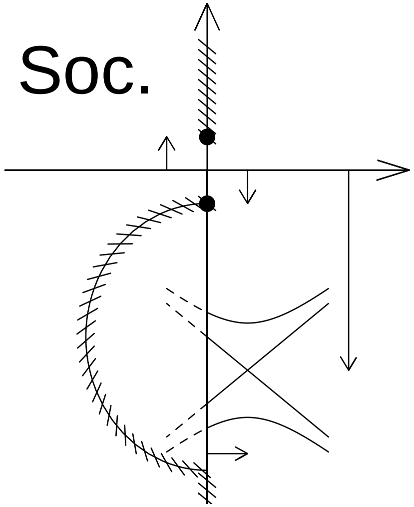
where $\sqrt{}$ denotes root with positive real part, i.e. branch-cuts are placed on $\mathrm{Im}(\alpha)$ axes.

- Homogeneous boundary conditions imply $C_2 = 0$ so that $v \to 0$ as $y \to \infty$.

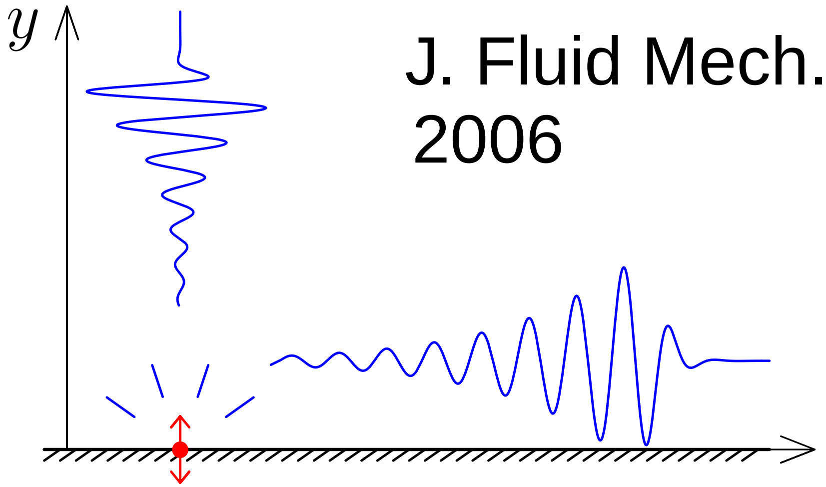
Left half-plane modes

My recent work for the rotating disk boundary layer:

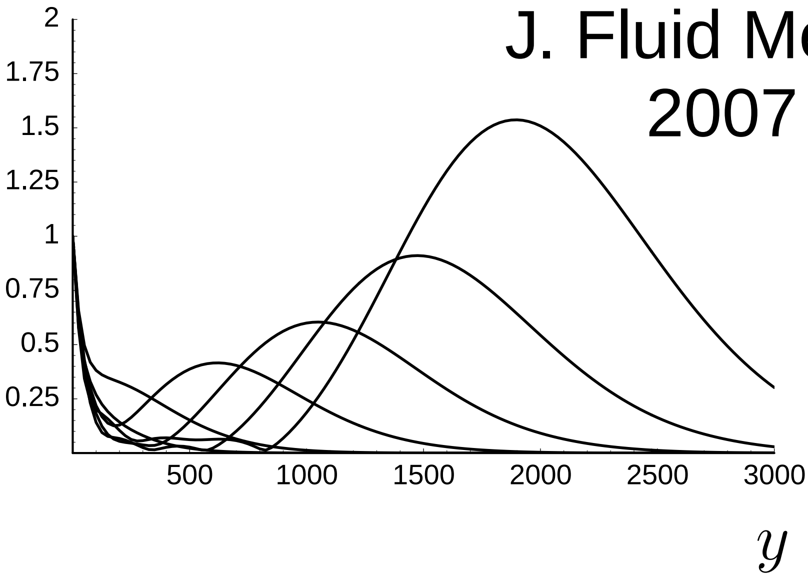
Proc. R. Soc. 2006

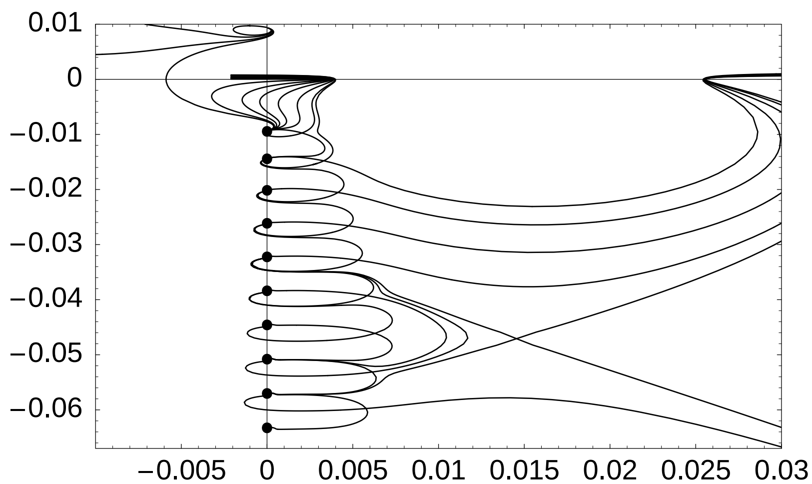


J. Fluid Mech. 2006



J. Fluid Mech. 2007





Confinement is destabilizing.

Through the Looking Glass

Twas brillig in the complex plane
of rolling hyperbolic hills,
with path of integration lain
through points where group speed stills.

Beware the Saddlewok my son!
which lurks at negative kay-r.
Its growth rate, hardly e'er outdone,
runs perpendicular.

So take numeric tool in hand
and bend the branch cut from the axe.
Reveal that tulgey curious land
where navely Briggs-Bers cracks.

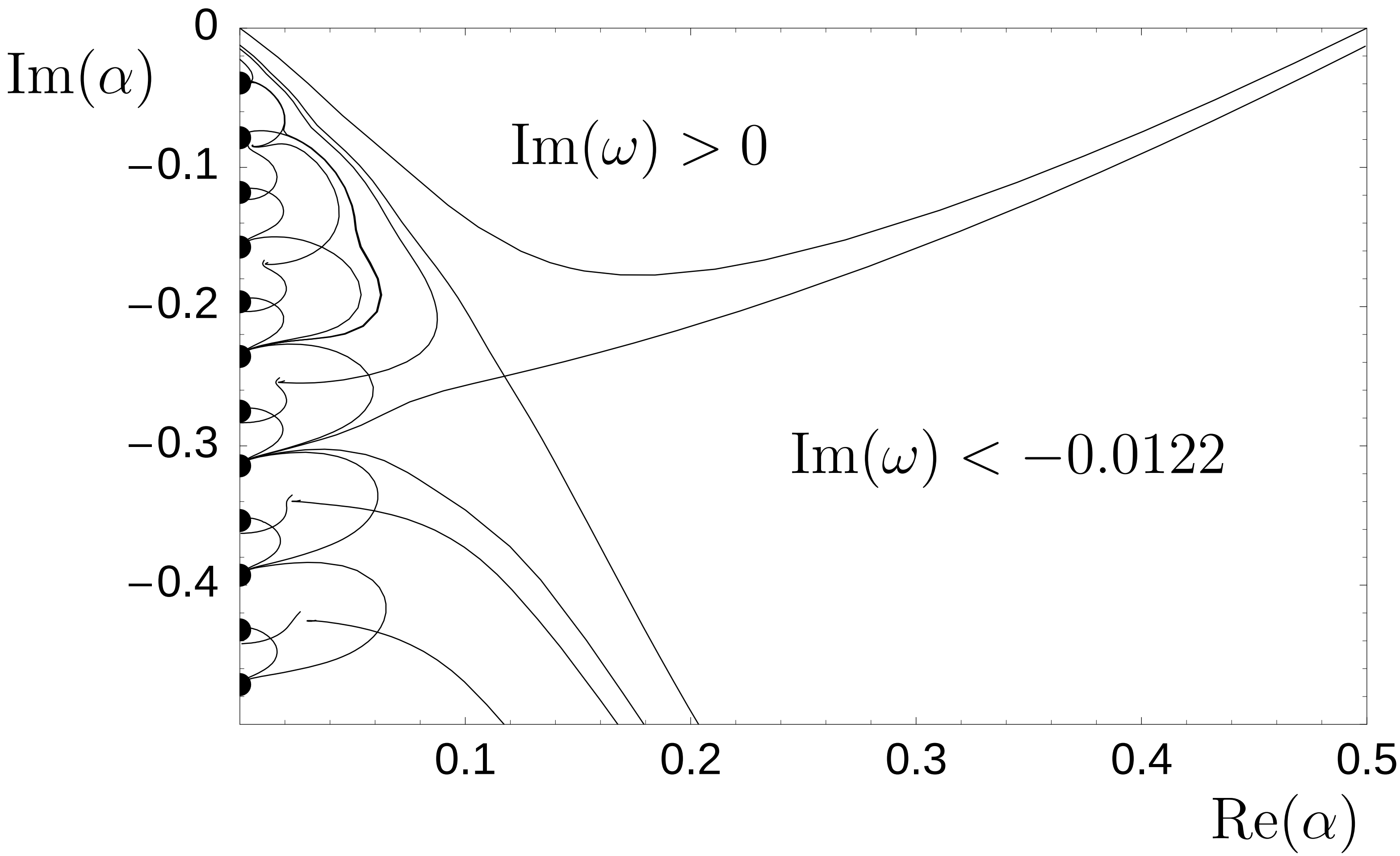
Thou'll findst the Saddlewok spits wrath:
"My mode blows up. Thou can'st touch me!"
Yet integrate 'long bended path,
I'll wage thou'd disagree.

The Saddlewok, you'd be surprised,
can be most easily explained:
its growing mode is localised;
the wavepacket contained.

Twas brillig in the complex plane
of rolling hyperbolic hills,
with path of integration lain
through points where group speed stills.

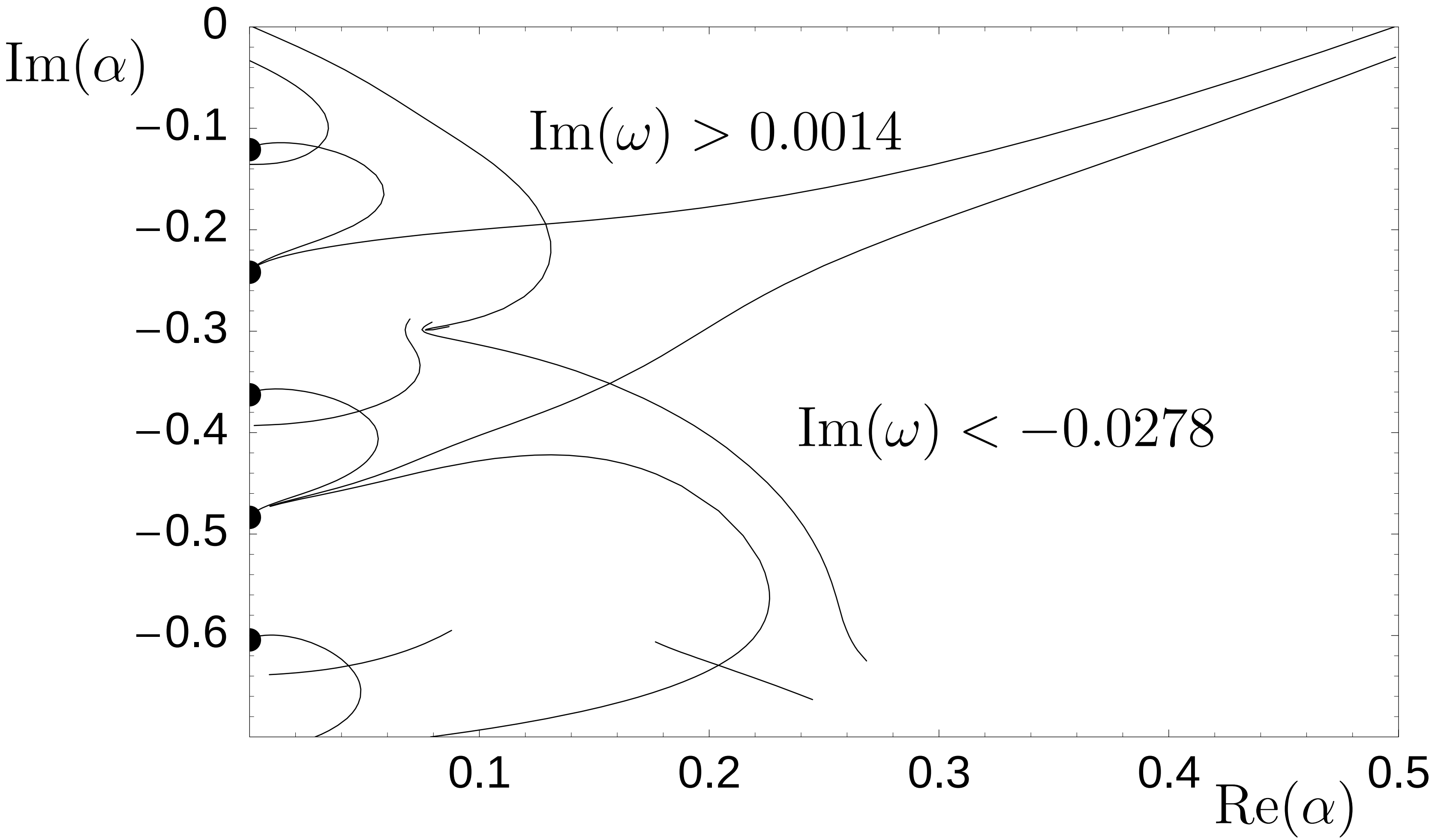
Confinement saddle points

An infinite number of saddle points are created, e.g. at $h = 40$, $r = 1.25$:



A more confined flow

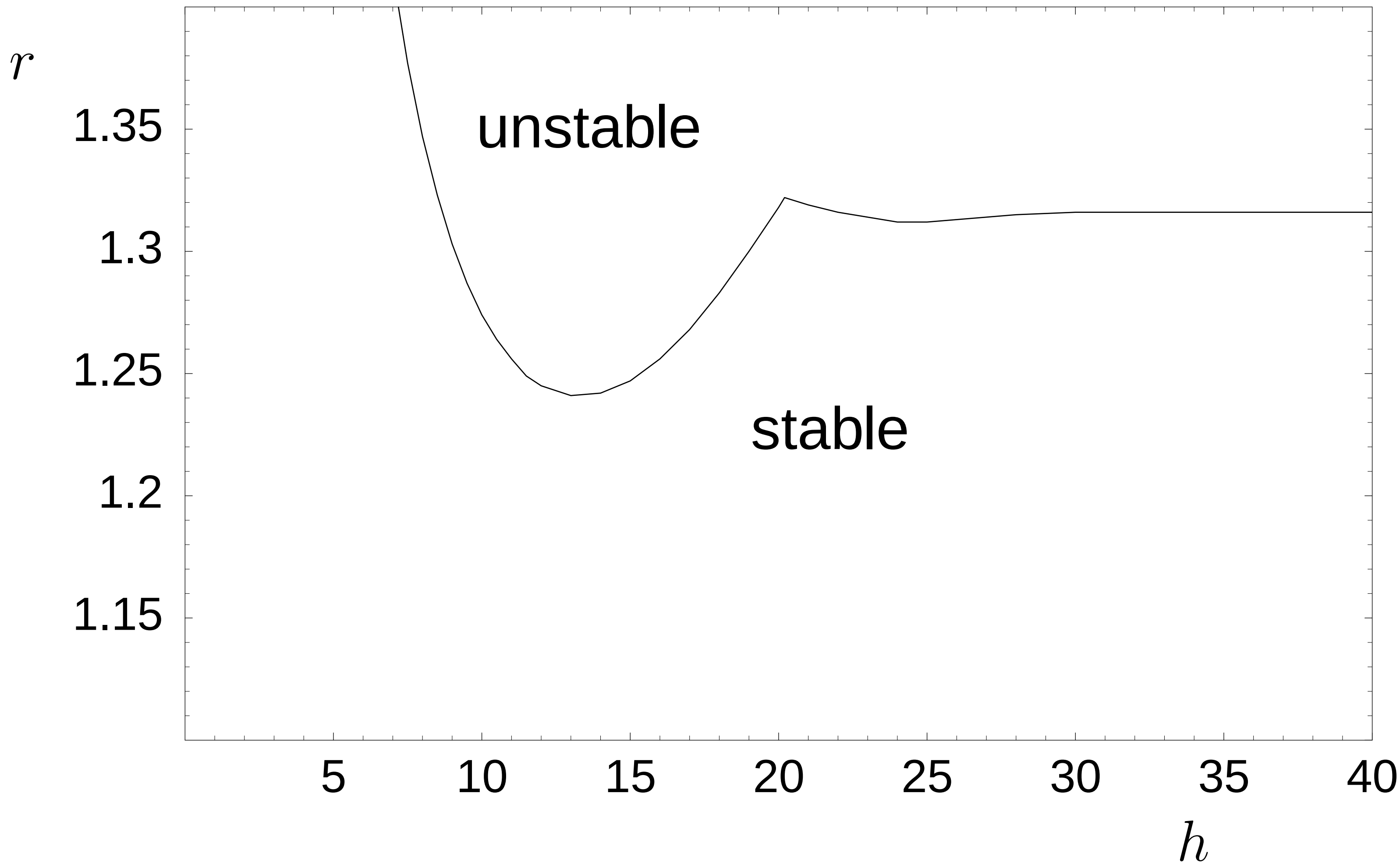
$h = 13$, $r = 1.25$



Flow has been made absolutely unstable by confinement.

Neutral curve for absolute instability

$$U = 1 + r\tanh(y/2)$$



Asymmetric confinement

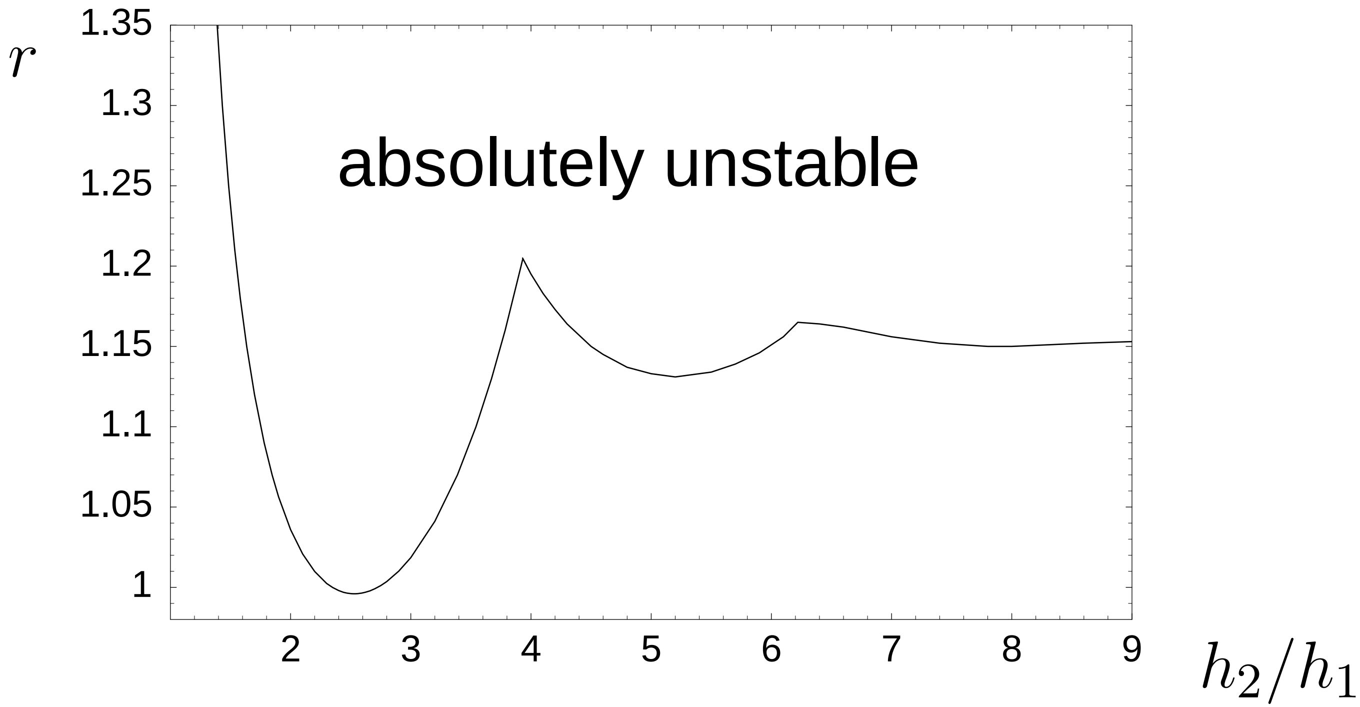
- Boundary conditions: $v = 0$ at plates at $y = h_1$ and $y = -h_2$.
- Leading-order long-wave dispersion relation is

$$\left[(c-1)^2 + r^2\right] \sinh \alpha(h_1 + h_2) = 2r(c-1) \sinh \alpha(h_2 - h_1).$$

- Symmetric confinement, $h_1 = h_2$, gives the unconfined K-H dispersion relation: $c = 1 \pm r\mathrm{i}$.
- But asymmetric confinement, $h_1 \neq h_2$, gives dispersion at leading order.

Asymmetrically confined long waves

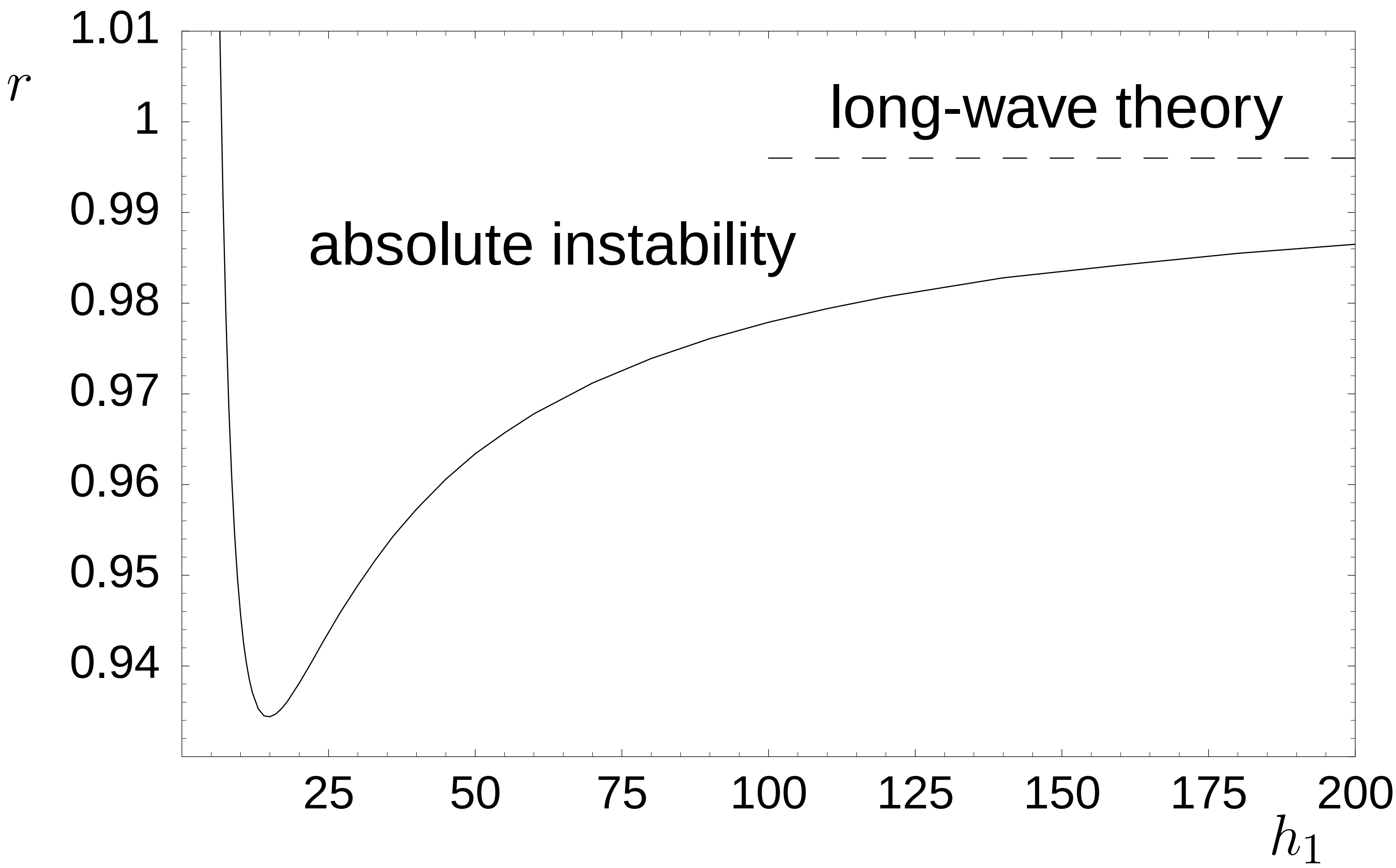
Neutral curve for absolute instability in long-wave limit:



- Tanh profile would be destabilized for $h_2/h_1 > 1.42$
- All profiles would give co-flow absolutely instability for $2.35 < h_2/h_1 < 2.72$.

Asymmetric confinement

Neutral curve for $h_2 = 2.53h_1$



Conclusions

- The effect of confinement on the absolute instability of mixing layers has been studied.
- Confinement creates additional saddle points.
- A confinement saddle can create absolute instability.
- Confinement saddle points can be described using long-wave theory.
- Asymmetric confinement can create co-flow absolute instability for any mixing layer profile: