Sprays: modelling versus experimentation

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MODELLING OF A VORTEX RING FLOW AT HIGH REYNOLDS NUMBERS

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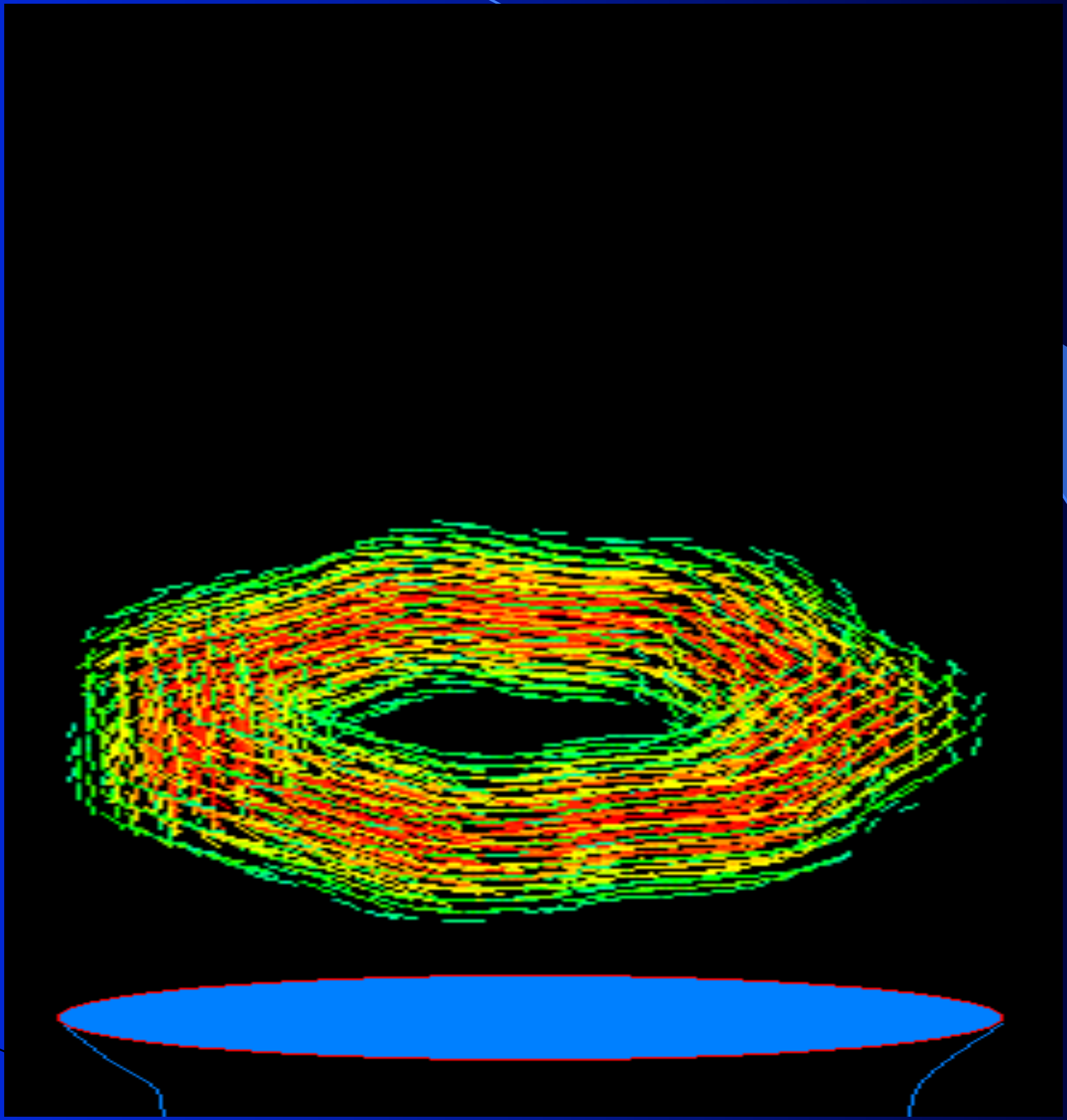
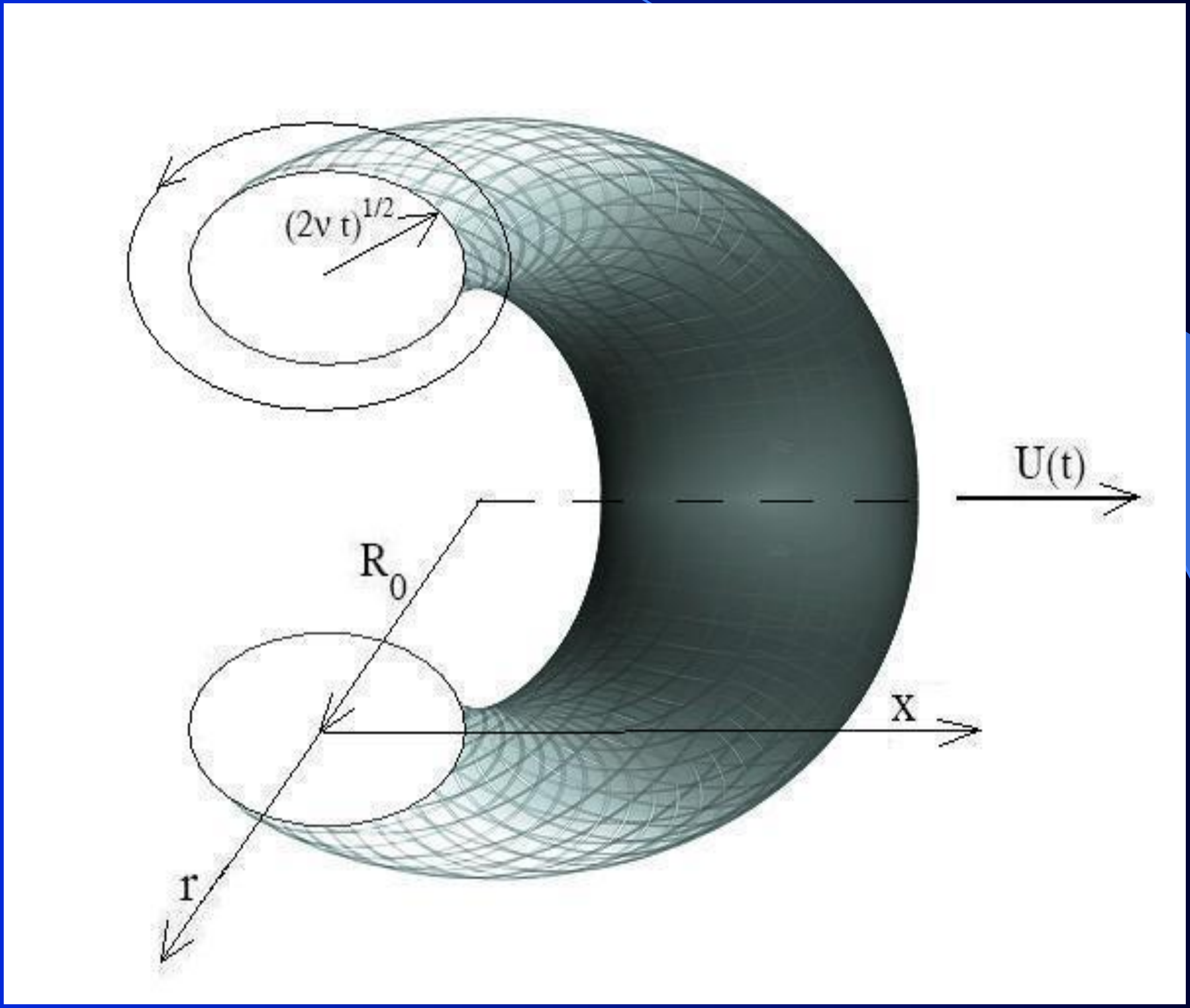
Virtual image of a vortex ring flow

Force acts impulsively



Schematic view of a vortex ring



Outline

- Basic solution
- Ring properties: translational velocity, energy, circulation, streamfunction
- Effect of the Reynolds number
- Turbulent vortex ring
- Applications: ‘optimal’ vortex ring formation, vortex ring-like structures in GDI engines

Formulation of the problem

$$\frac{\partial \varsigma}{\partial t}+\frac{\partial}{\partial r}(v\varsigma)+\frac{\partial}{\partial x}(u\varsigma)=\nu\left[\frac{\partial^2\varsigma}{\partial x^2}+\frac{\partial^2\varsigma}{\partial r^2}+\frac{1}{r}\frac{\partial\varsigma}{\partial r}-\frac{\varsigma}{r^2}\right]$$

$$u=\frac{1}{r}\frac{\partial\Psi}{\partial r}+U(t)$$

$$,v=-\frac{1}{r}\frac{\partial\Psi}{\partial x},$$

$$U(t)=\frac{dx_0(t)}{dt}$$

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{\partial^2\Psi}{\partial x^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r}=-r\,\varsigma$$

$$\Psi(0,x)=\varsigma(0,x)=0,$$

$$(x^2+r^2)^{1/2}\to\infty:\Psi,\varsigma\to 0$$

$$M=\frac{I}{\rho}$$

$$I=\pi\rho\int_0^{\infty}\int_{-\infty}^{\infty}r^2\varsigma\,dx\,dr$$

$$\sigma=\frac{r}{\ell},\eta=\frac{x\text{-}x_0(t)}{\ell},\quad \Phi=\frac{\Psi}{\varsigma_0\ell^3},\omega=\frac{\varsigma}{\varsigma_0},$$

$$\ell=\sqrt{2\nu t},\varsigma_0=A(M,\nu,R_0)\,t^{-a}$$

$$\tau=\frac{R_0}{\ell}$$

ring-to-core radius

Basic solution

$$-a\,\omega-\sigma\frac{\partial\omega}{\partial\sigma}-\eta\frac{\partial\omega}{\partial\eta}-\tau\frac{\partial\omega}{\partial\tau}+\boldsymbol{Re}\left[\frac{\partial}{\partial\sigma}\left(-\frac{1}{\sigma}\frac{\partial\Phi}{\partial\eta}\omega\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}\omega\right)\right]$$

$$=\frac{\partial^2\omega}{\partial\sigma^2}+\frac{\partial^2\omega}{\partial\eta^2}+\frac{1}{\sigma}\frac{\partial\omega}{\partial\sigma}-\frac{\omega}{\sigma^2}$$

$$\boldsymbol{Re}=\varsigma_0\ell^2/\nu$$

$$\omega(\sigma,\eta,\tau;\boldsymbol{Re})=\omega_1(\sigma,\eta,\tau)+\boldsymbol{Re}\,\omega_2(\sigma,\eta,\tau)+..$$

$$\Phi(\sigma,\eta,\tau;\boldsymbol{Re})=\Phi_1(\sigma,\eta,\tau)+\boldsymbol{Re}\,\Phi_2(\sigma,\eta,\tau)+..$$

$$\omega_1=\exp\left(-\frac{1}{2}\left(\sigma^2+\eta^2+\tau^2\right)\right)\mathrm{I}_1(\sigma\tau),$$

$$\varsigma_0=\frac{2M}{(4\pi\nu t)^{3/2}R_0}$$

Limiting cases

short times $t \to 0$ → *identical to the vorticity distribution obtained using the asymptotic analysis for rings at the initial stage (Chi-Tzung Wang et al., 1994)*

$$\varsigma = \frac{M}{4\pi^2 \nu t R_0^2} \exp\left(-\frac{(x - x_0(t))^2 + (r - R_0)^2}{4\nu t}\right)$$

long times $t \to \infty$ → *identical to the vorticity distribution obtained for rings at the decaying stage (Phillip's, 1956)*

$$\varsigma = \frac{Mr}{16\pi^{3/2} (\nu t)^{5/2}} \exp\left(-\frac{(x - x_0(t))^2 + r^2}{4\nu t}\right)$$

The solution at the initial stage has the Gaussian form.

The Fourier- Hankel integral transform

$$\omega = exp\left(-\frac{1}{2}\left(\sigma^2 + \eta^2 + \tau^2\right)\right) I_1(\sigma\tau)$$

$$\Rightarrow$$

$$\bar{\omega} = exp\left(-\frac{\mu^2 + \alpha^2}{2}\right) J_1(\tau\mu)$$

$$\frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \eta^2} - \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma} = -\sigma\omega$$

$$\Rightarrow$$

$$\bar{f} = \frac{exp\left(-\frac{\mu^2 + \alpha^2}{2}\right)}{\mu^2 + \alpha^2} J_1(\tau\mu)$$

$$f = \Phi / \sigma$$

Kinetic energy and translational speed

$$E = \frac{\sqrt{\pi}\,\Gamma_0^2 R_0^4}{48\sqrt{2}\,(\nu t)^{3/2}}\ {}_2F_2\left(\frac{3}{2},\frac{3}{2};\frac{5}{2},3;-\frac{R_0^2}{2\nu t}\right)$$

$$U = \frac{\Gamma_0 R_0^2}{96\sqrt{2\pi}\,(\nu t)^{3/2}}\left\{ {}_2F_2\left(\frac{3}{2},\frac{3}{2};\frac{5}{2},3;-\frac{R_0^2}{2\nu t}\right) - \frac{36}{5}\,{}_2F_2\left(\frac{3}{2},\frac{5}{2};2,\frac{7}{2};-\frac{R_0^2}{2\nu t}\right) + \frac{72\nu t}{R_0^2}\exp\left(-\frac{R_0^2}{2\nu t}\right) I_1\left(\frac{R_0^2}{4\nu t}\right)\right\}$$

Limit $\nu t << R_0^2$

$$U \to U_s = \frac{\Gamma_0}{4\pi R_0}\left\{\ln\left(\frac{4R_0}{\sqrt{\nu t}}\right) - 0.558 + O\left[\frac{\sqrt{2\nu t}}{R_0}\ln\left(\frac{2\nu t}{R_0^2}\right)\right]\right\}$$

$$E \to E_s = \frac{\Gamma_0^2 R_0}{2}\left\{\ln\left(\frac{4R_0}{\sqrt{\nu t}}\right) - 2.058 + O\left[\frac{\sqrt{2\nu t}}{R_0}\ln\left(\frac{2\nu t}{R_0^2}\right)\right]\right\}$$

These equations are identical with those reported by Safffman (1970); (see Fukumoto & Kaplanski, 2007, submitted for publication).

Limit $\nu t >> R_0^2$

$$U \rightarrow U_{RC} = 0.0037037594 \frac{I}{(\nu t)^{3/2}}$$

This equation is identical with the one reported by Rott & Cantwell (1993); (see Kaplanski & Rudi,2005).

$$E \rightarrow E_{RC} = 0.00264557 \frac{I^2}{(\nu t)^{3/2}}$$

Streamfunction

$$(f = \Psi / r) \leftarrow \overline{f}$$

$$\Psi = \frac{\Gamma_0 R_0 r}{4} \int_0^\infty \left[e^{px} erfc\left(\frac{2p\nu t + x}{2\sqrt{\nu t}}\right) + e^{-px} erfc\left(\frac{2p\nu t - x}{2\sqrt{\nu t}}\right) \right] J_1(pR_0) J_1(pr)\, dp$$

Limit

$$\nu t << R_0^2$$

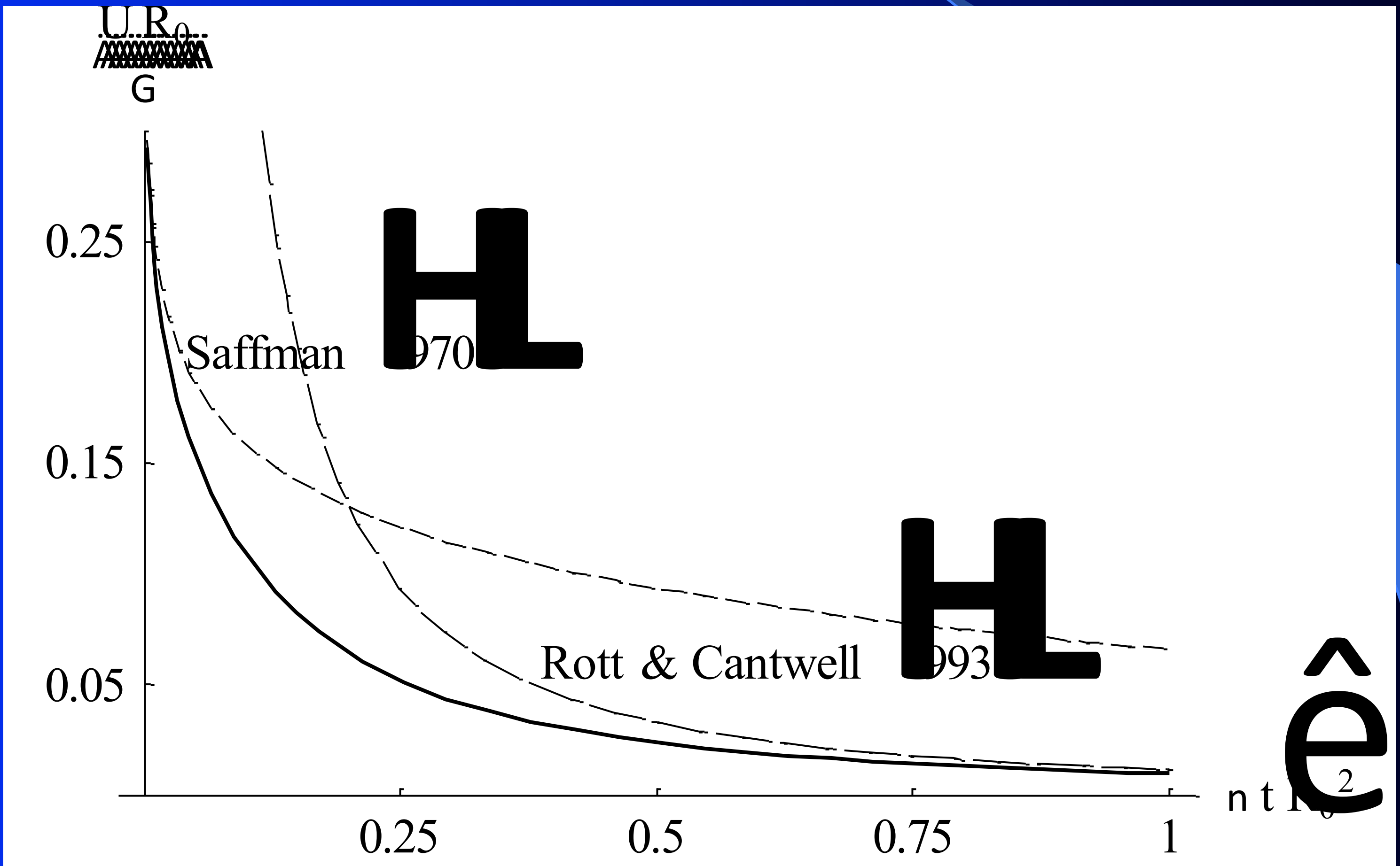
$$\Psi \approx \frac{\Gamma_0 R_0 r}{4} \int_0^\infty e^{p|x|} J_1(pR_0) J_1(pr)\, dp$$

Lamb, 1932

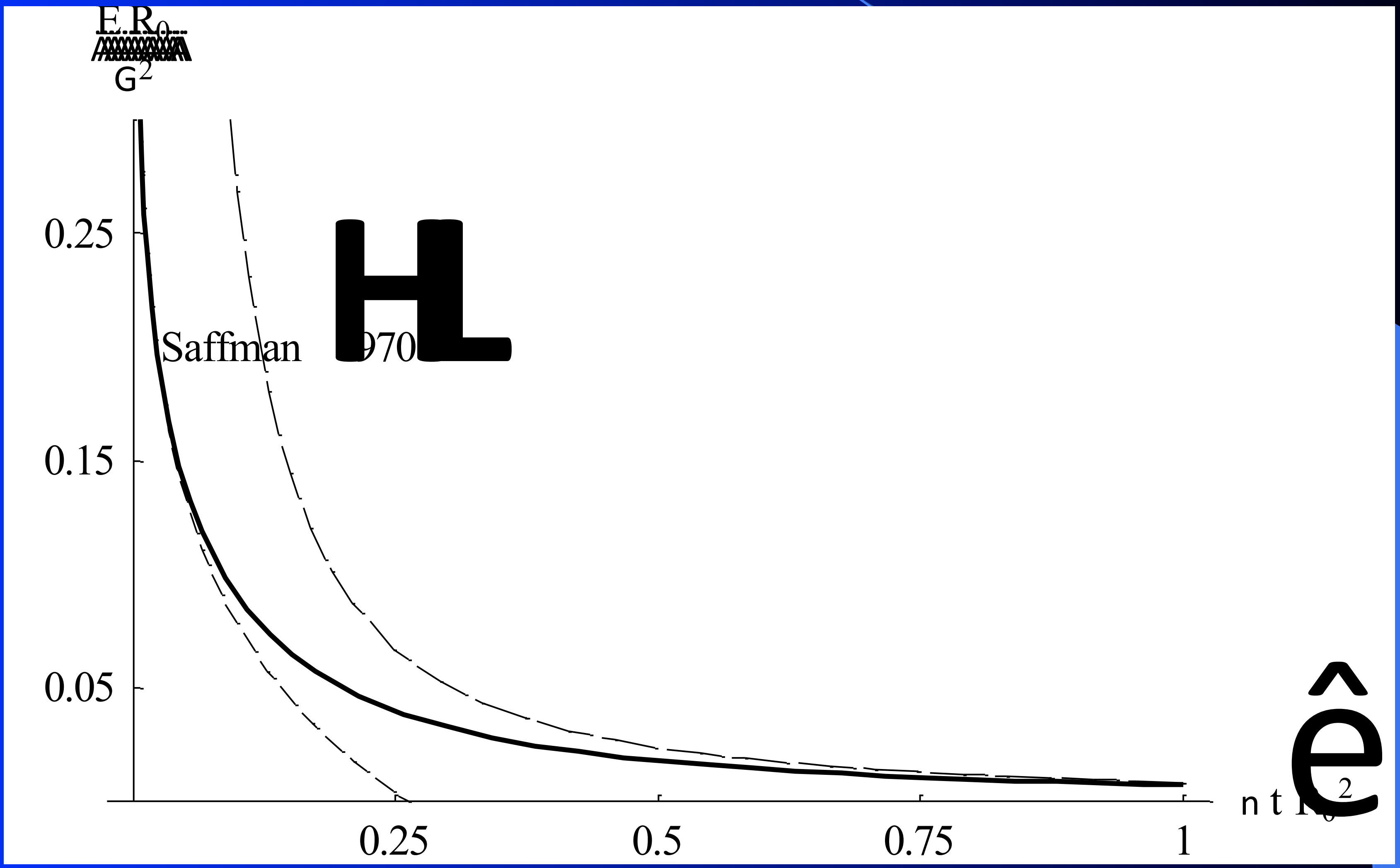
Circulation

$$\Gamma = \Gamma_0 \left\{ 1 - \exp\left(-\frac{R_0^2}{4\nu t}\right) \right\}$$

Comparison of the results (speed)

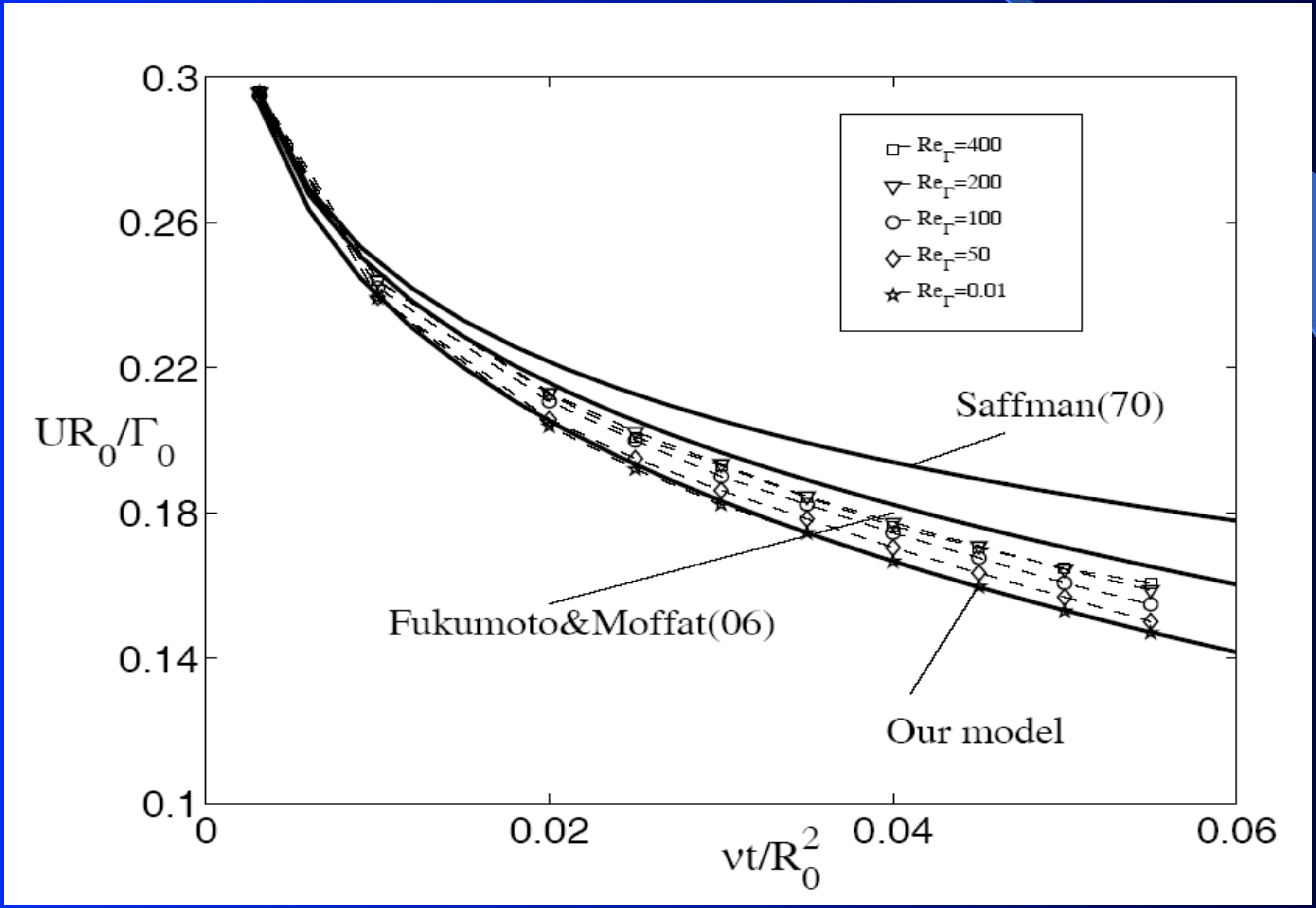


Comparison of the results (kinetic energy)



Effect of the Reynolds number (initial stage)

Numerical simulations (S.Stanaway, B.J. Cantwell and P.R. Spalart, NASA Technical Memorandum 101041 (1988)) versus analytical results



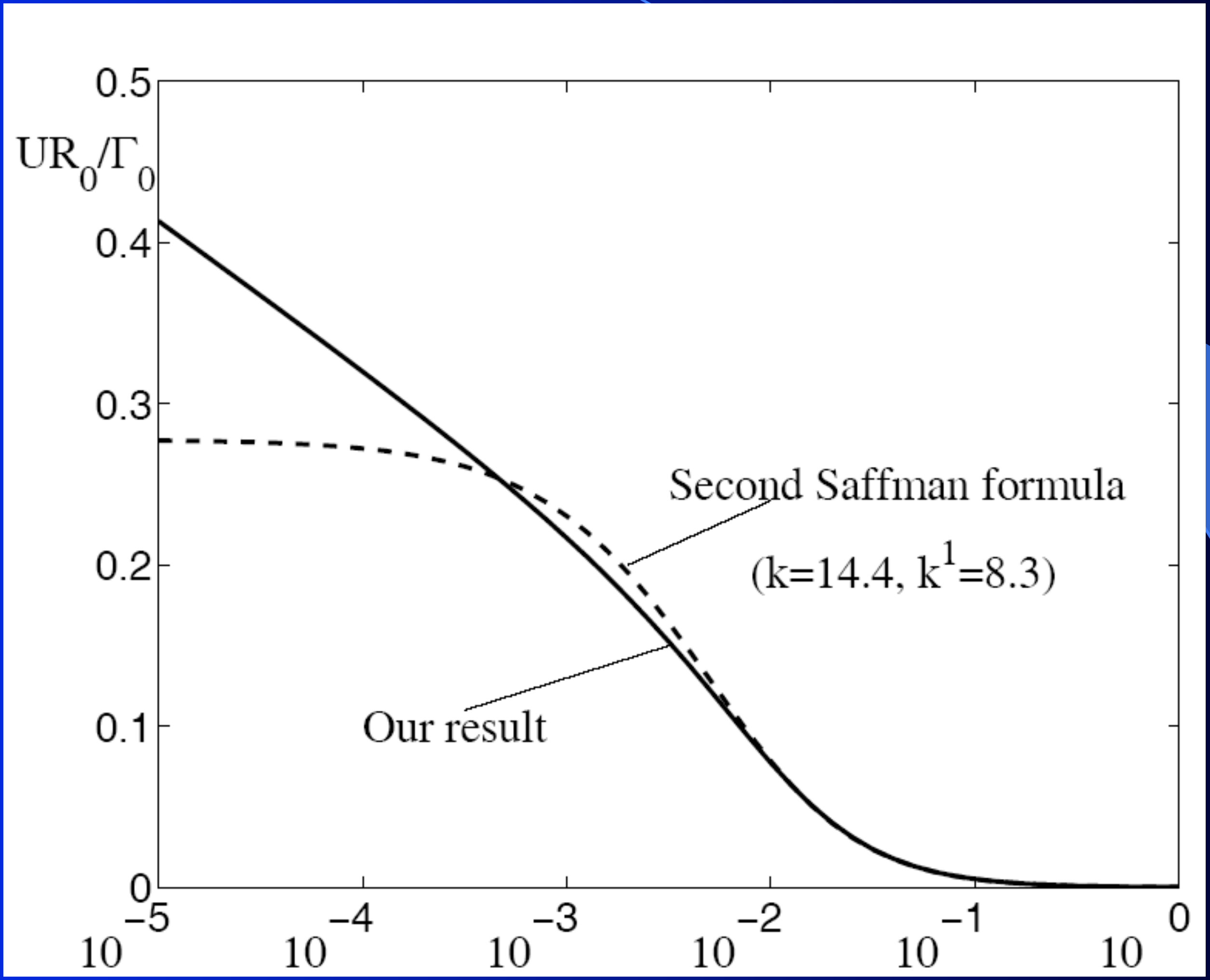
Fukumoto & Kaplanski, 2007, submitted for publication.

Second Saffman's formula for $\nu t \approx R_0^2$

$$U = \frac{\Gamma_0}{R_0}\frac{16\pi}{k}(1 + 16k' t^*)^{-3/2},$$

wher $t^* = \nu t / (4D_0^2)$, k and k′ are tunable constants .

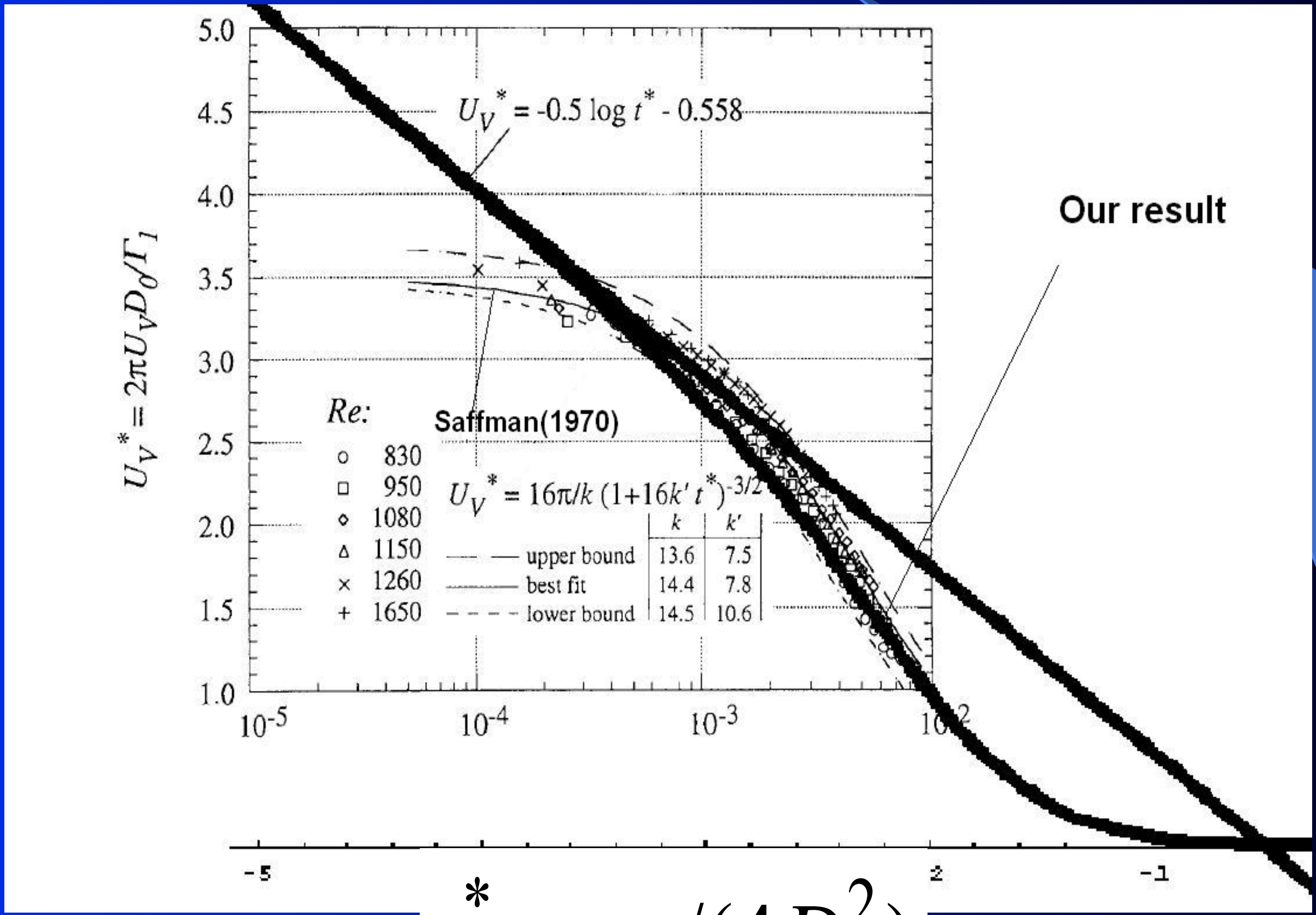
Comparison of the results



$t^{*} = \nu t / (4 D_0^2)$

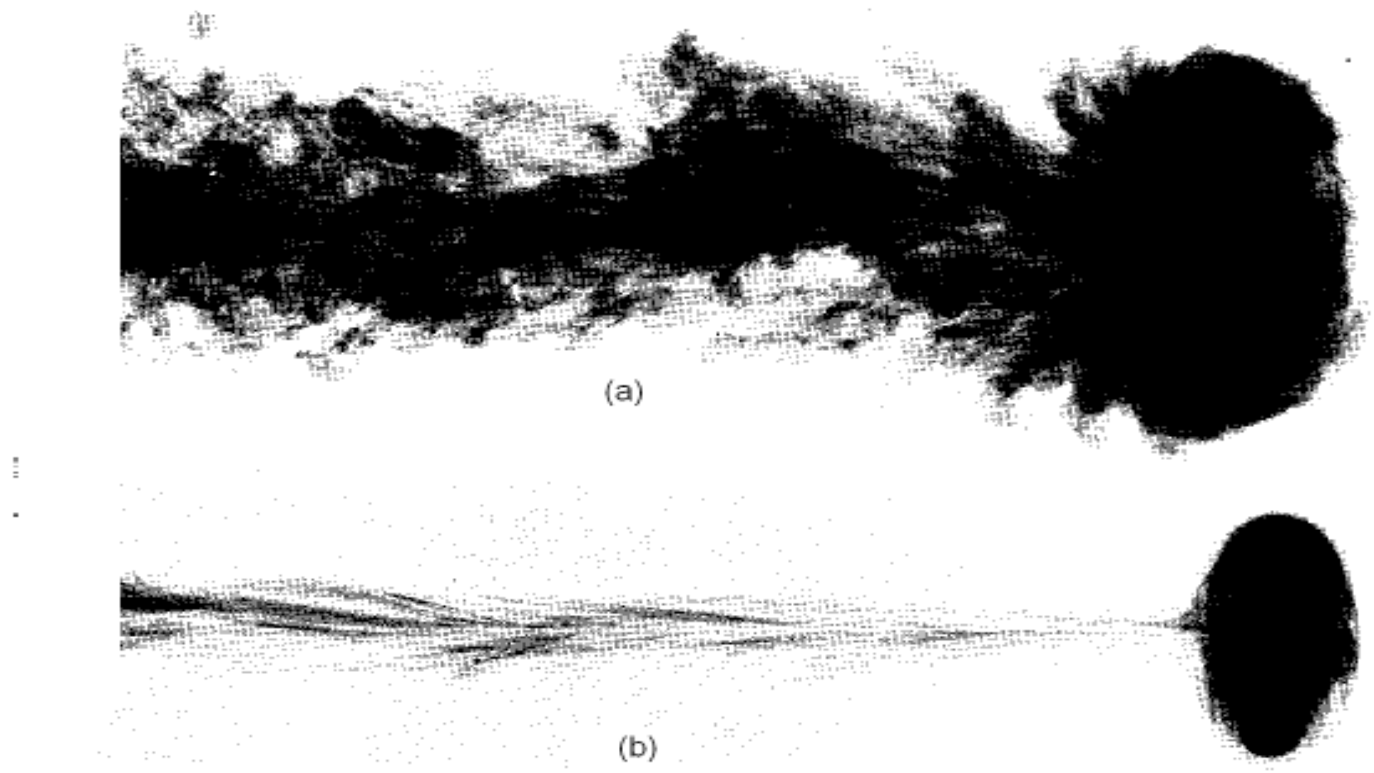
Fukumoto&Kaplanski, 2007, submitted for publication.

Comparison the results with experimental data (see Weigand & Gharib,1997)



$$t^* = \nu t/(4D_0^2)$$

Turbulent and laminar vortex rings produced by an impulsive force (Glezer & Coles, 1990). Initial Reynolds number (a) 27000, (b) 7500.



- In a gross sense the overall mean field in a turbulent flow tends to behave somewhat like a very viscous constant Reynolds number flow.

Brian J. Cantwell (Introduction to Symmetry Analysis, Cambridge Texts in Applied Mathematics, 2002)

Arbitrary scales

$$\varsigma_0 = A(M, \nu_*, R_0)\, t^{-\lambda}$$

$$\ell \approx t^b, \nu_* = \ell \ell'$$

$$\tau = \frac{R_0}{\ell},$$

$$-\frac{\lambda \ell^2}{t\nu_*}\omega - \frac{\ell \ell'}{\nu_*}\left(\sigma \frac{\partial \omega}{\partial \sigma} - \eta \frac{\partial \omega}{\partial \eta} - \tau \frac{\partial \omega}{\partial \tau}\right) + \mathrm{Re}\left[\frac{\partial}{\partial \sigma}\left(-\frac{1}{\sigma}\frac{\partial \Phi}{\partial \eta}\omega\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma}\omega\right)\right] =$$

$$= \left[\frac{\partial^2 \omega}{\partial \eta^2} + \frac{\partial^2 \omega}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial \omega}{\partial \sigma} - \frac{\omega}{\sigma^2}\right]$$

$$\mathrm{Re} = \varsigma_0 \ell^2 / \nu_*$$

The solution exists when:

$$-\frac{\lambda \ell^2}{t\nu_*} = -\frac{\lambda}{b} = -3$$

$$A\frac{\sqrt{2}\left(\pi \nu_*\right)^{3/2} R_0}{b^{3/2}} t^{-\lambda+3b} = M$$

$$\lambda = 3b$$

Limit $\tau = R_0 / \ell \to 0$

$$\varsigma_0 = A(M, v_*) t^{-\lambda}$$

$$\ell \approx t^b, v_* = \ell \ell'$$

$$-\frac{\lambda \ell^2}{t v_*}\omega - \frac{\ell \ell'}{v_*}\left(\sigma \frac{\partial \omega}{\partial \sigma} - \eta \frac{\partial \omega}{\partial \eta}\right) + \mathrm{Re}\left[\frac{\partial}{\partial \sigma}\left(-\frac{1}{\sigma}\frac{\partial \Phi}{\partial \eta}\omega\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma}\omega\right)\right] = \ldots$$

$$= \left[\frac{\partial^2 \omega}{\partial \eta^2} + \frac{\partial^2 \omega}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial \omega}{\partial \sigma} - \frac{\omega}{\sigma^2}\right]$$

The solution exists when

$$\lambda = 4b$$

'Turbulent ' scales-

$$b = 1/4, \lambda = 4/4, \ell = \alpha M^{1/4} t^{1/4}$$

$$\varsigma = \varsigma_0 \left(\frac{r}{\alpha M^{1/4} t^{1/4}} \exp\left(-\frac{r^2 + x^2}{2\alpha^2 M^{1/2} t^{1/2}}\right) = \varsigma_0 r_1 \exp(-(r_1^2 + x_1^2)/2),\right.$$

$$\psi = \frac{\sqrt{\pi}\varsigma_0 \ell^3}{\sqrt{2}} [erf(s) - \frac{2s}{\sqrt{\pi}} \exp(-s^2)] \frac{r_1^2}{(r_1^2 + x_1^2)^{3/2}},$$

$$s = (r_1^2 + x_1^2)^{1/2}, \varsigma_0 = \frac{t^{-1}}{2\sqrt{2}\alpha^4 \pi^{3/2}}, r_1 = r/\ell, x_1 = x/\ell,$$

$$\mathrm{Re} = \frac{4}{\alpha^4 (2\pi)^{3/2}}$$

Phillip's self-similar solution (cf. the Lugovtsov model (1970))

Translational speed of the turbulent vortex ring

$$U = \frac{\int_0^{\infty}\int_{-\infty}^{\infty}(\Psi + 6xrv)\varsigma\,dx\,dr}{2\int_0^{\infty}\int_{-\infty}^{\infty} r^2\varsigma\,dx\,dr}$$

$$U = \frac{0.0105}{\alpha^3}\left(\frac{I}{\rho}\right)^{1/4}(t - t_0)^{-3/4}$$

$$U = \frac{1}{4\pi^{1/4}}\left(2\frac{I}{\rho}\right)^{1/4}(t - t_0)^{-3/4}$$

(Afanasyev *et. al*., 2004)

$$\alpha = 0.77$$

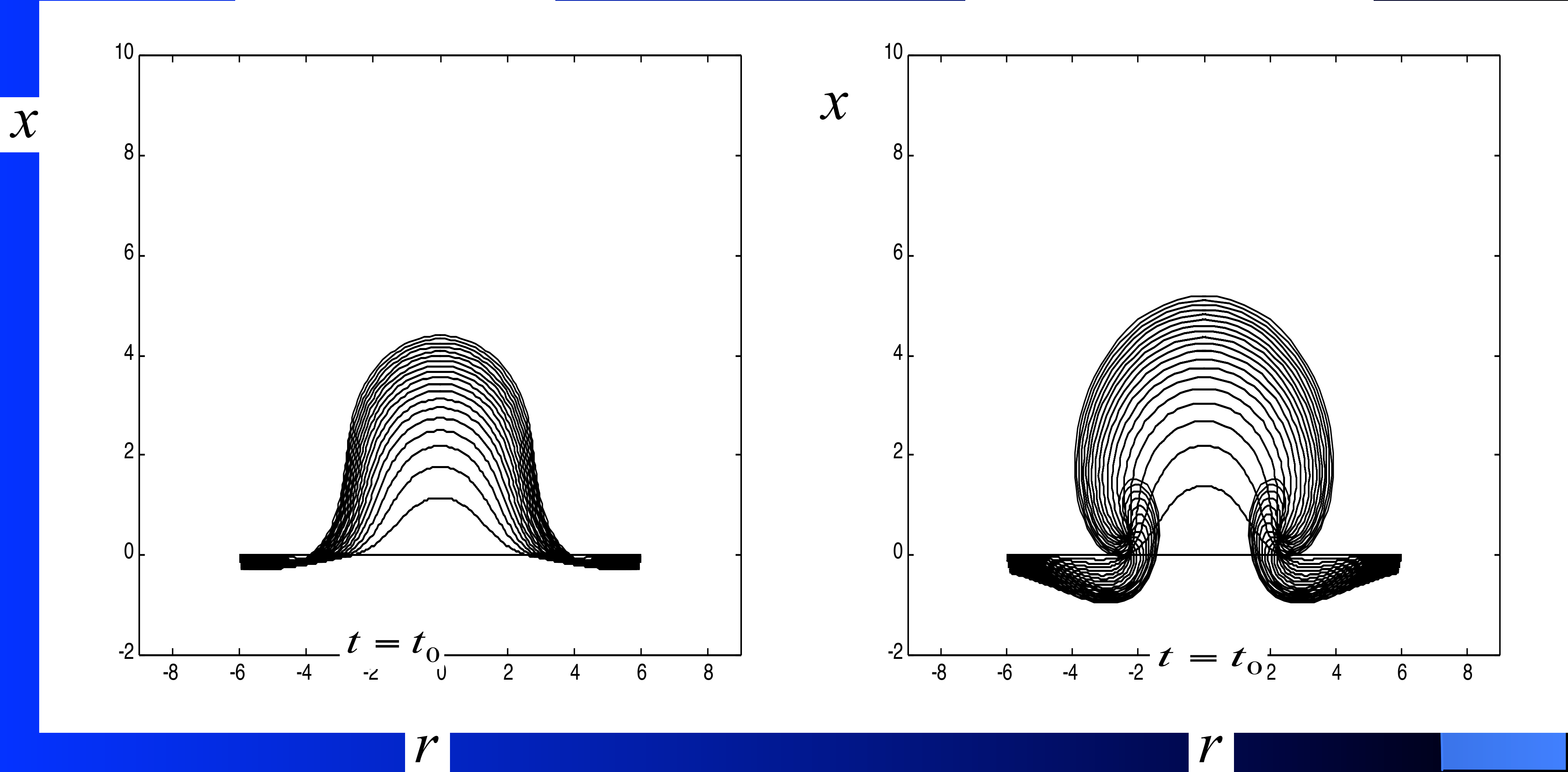
Effect of α

Timelines of the turbulent vortex ring for I/ρ=1

$$\frac{dx}{dt} = u_t + U,$$

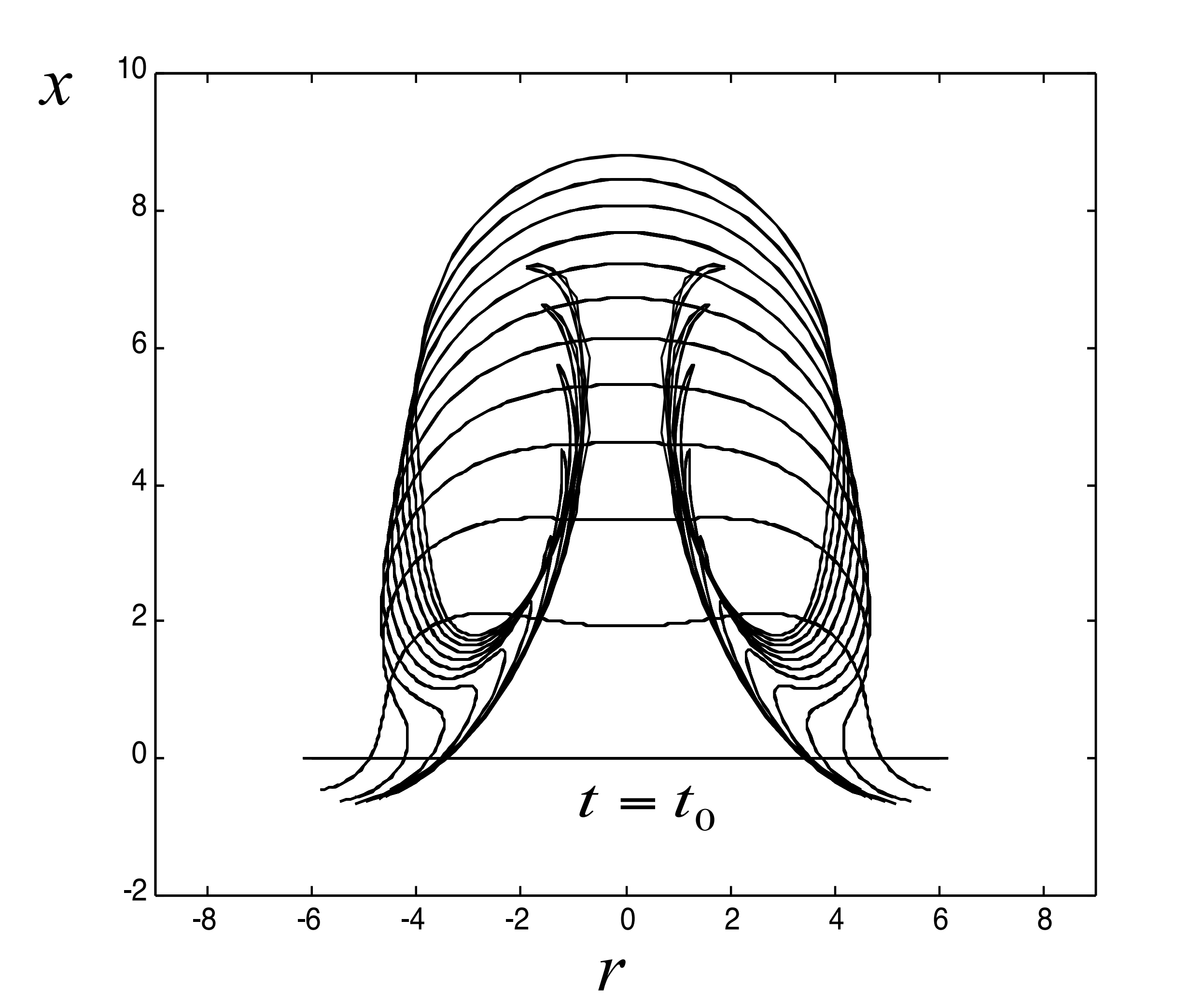
$$\frac{dr}{dt} = v_t.$$

$\alpha = 0.77$ $\alpha = 0.1187$



Timelines of the laminar vortex ring for

$$\tau_0 = \frac{R_0}{\sqrt{2\nu t_0}} = 5, \nu = 0.01, R_0 = 1.$$

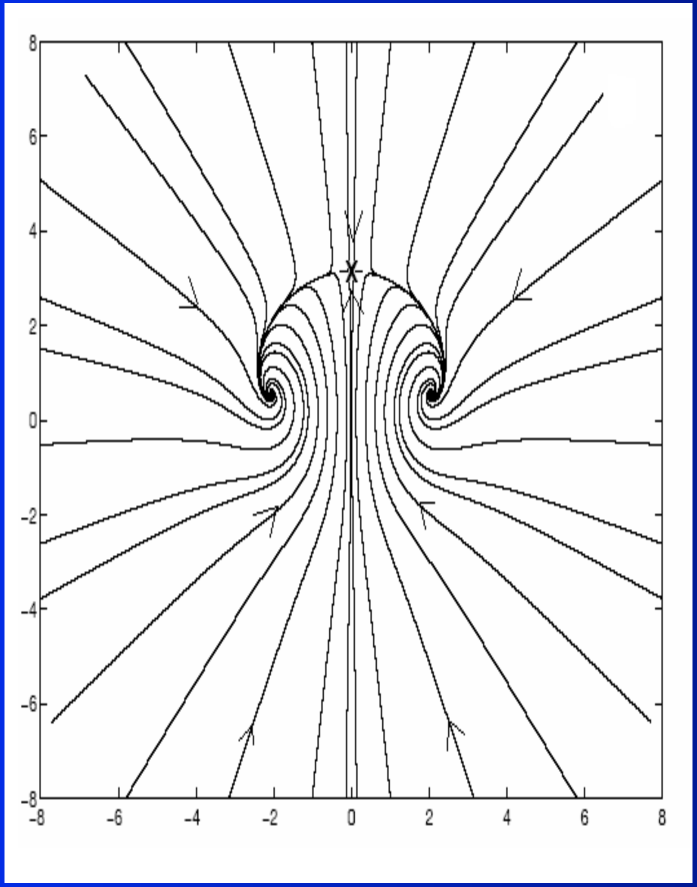


Entrainment diagrams for a turbulent ring

$$\frac{d\eta}{ds} = -\frac{\eta}{4} + \operatorname{Re} u_t,$$

$$\frac{d\sigma}{ds} = -\frac{\sigma}{4} + \operatorname{Re} v_t, s = \ln(t).$$

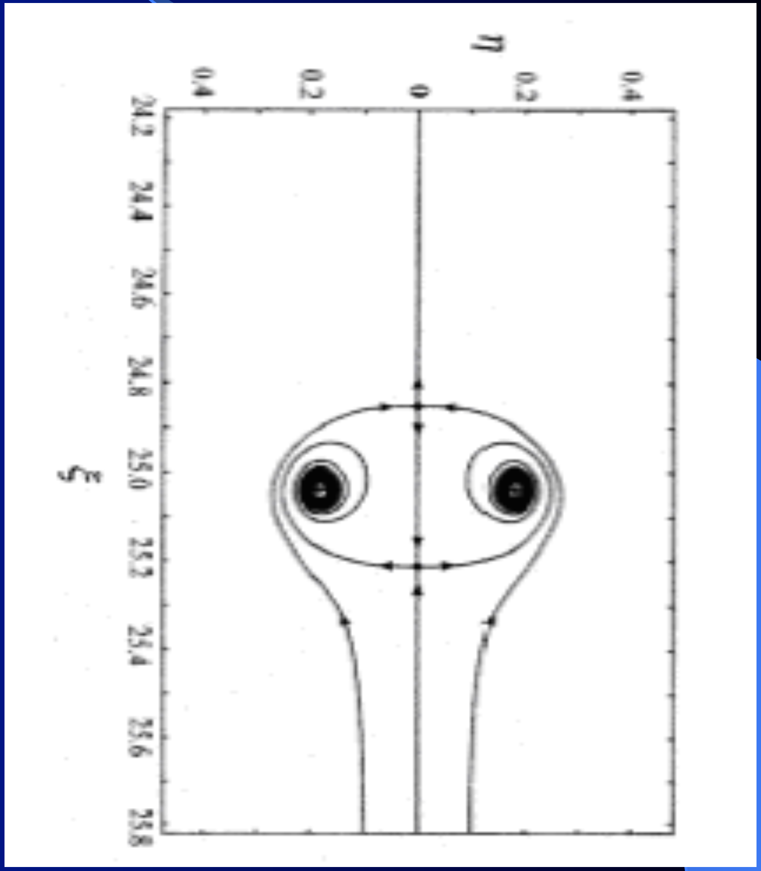
η



σ

Calculations

η



σ

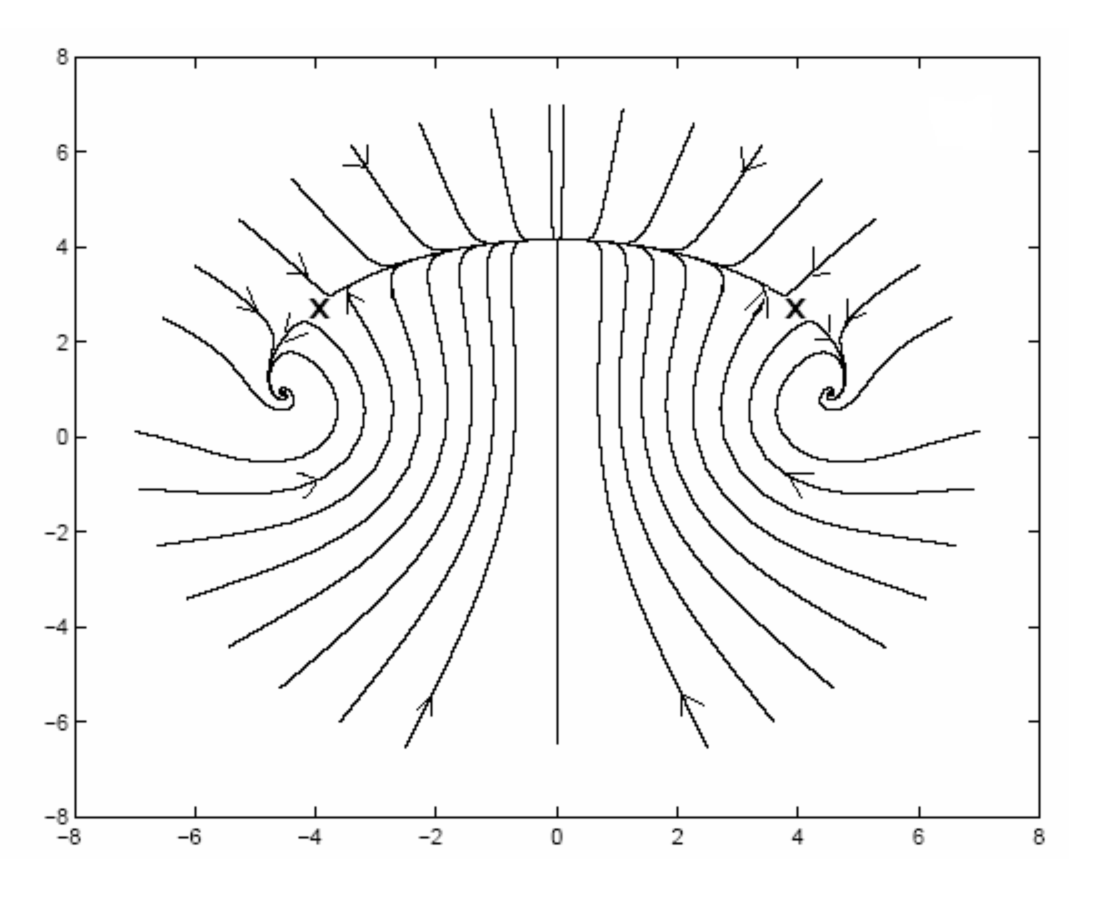
Glezer & Coles, 1990

Entrainment diagram for a laminar ring

$$\frac{d\eta}{ds} = -\frac{\eta}{2} + \mathrm{Re}_0\, \tau u_t,$$

$$\frac{d\sigma}{ds} = -\frac{\sigma}{2} + \mathrm{Re}_0\, \tau\, v_t,\ s = \ln(t).$$

η



σ

Schematic view of vortex ring generator (Gharib *et al.*, 1998)

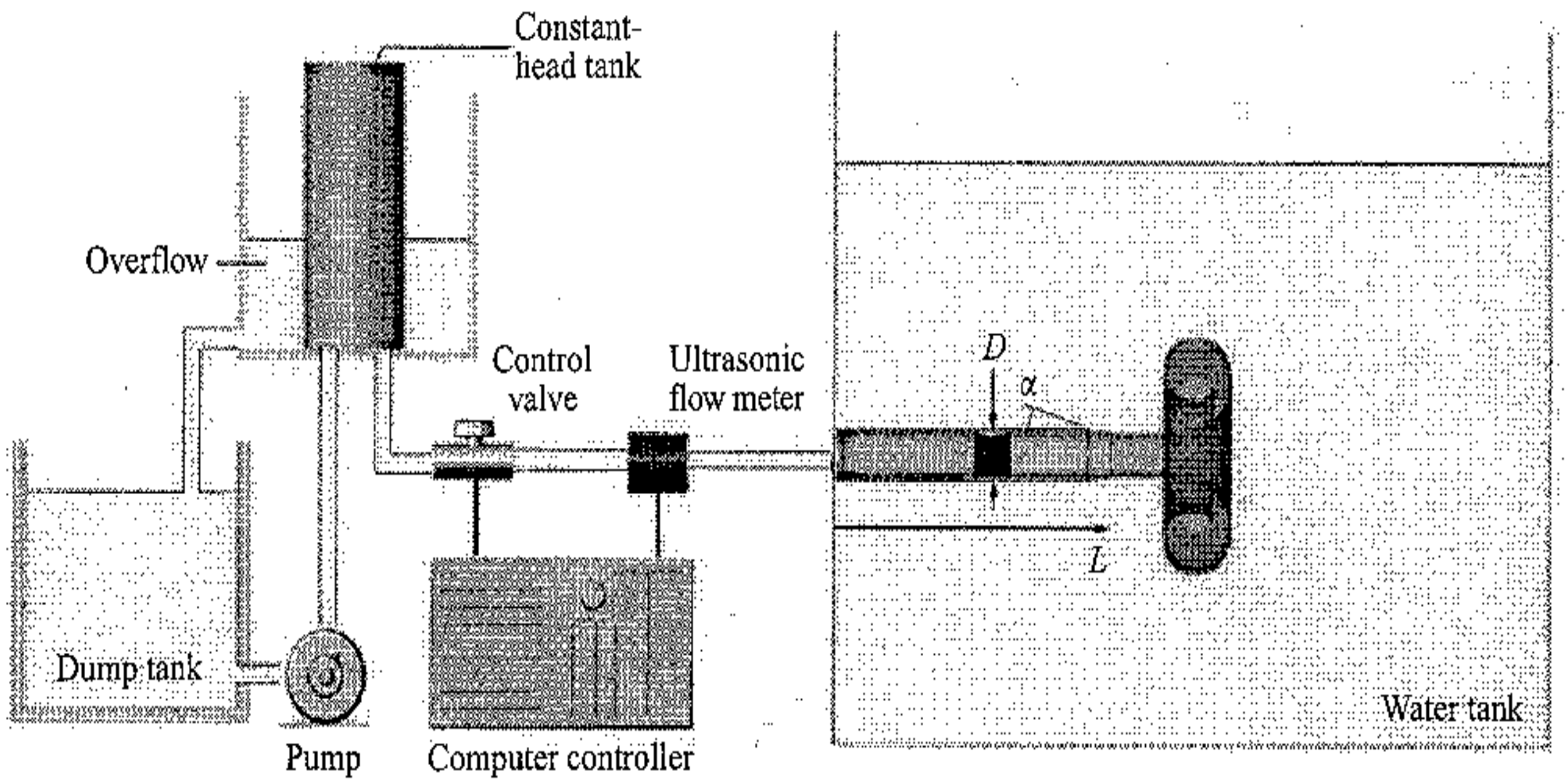
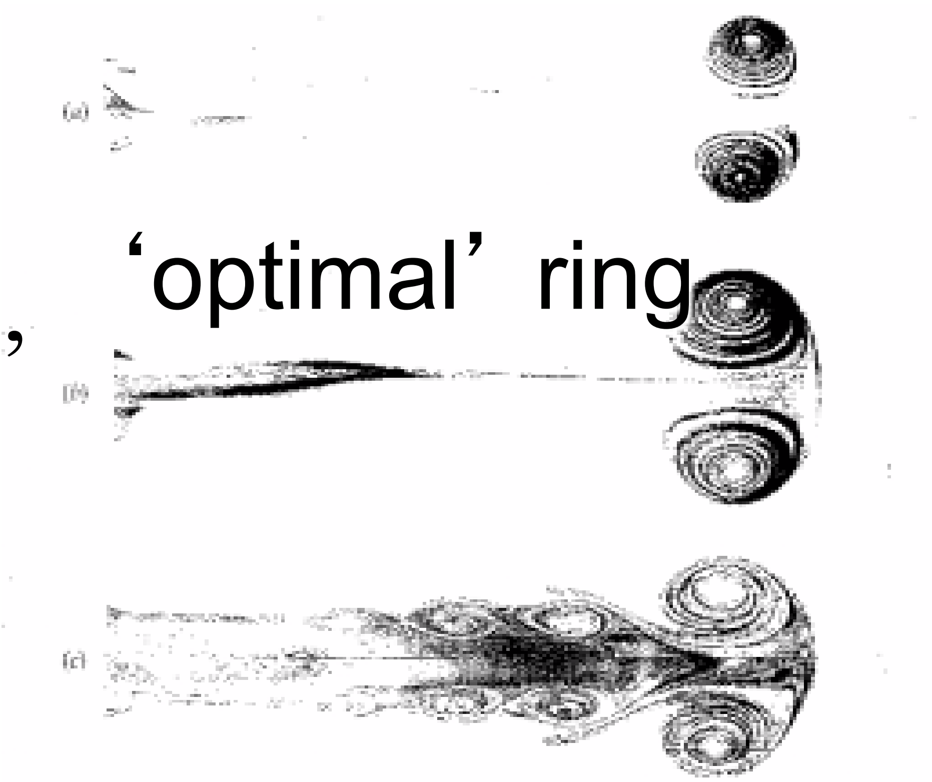


FIGURE 1. General schematic of vortex ring generator.

Formation stage (Gharib *et al.*,1998)

$E_{lim} \approx 0.3$,
$\Gamma_{lim} \approx 2.0$

‘optimal’ ring



L/D<4

L/D≈4

L/D>4

Estimate of the 'formation' number (kinematic approach, Shusser&Gharib, 2000)

Equation for τ at the pinch–off, based on the slug model

$$\alpha(\tau) = \frac{B(\tau)}{2N(\tau)\sqrt{\pi}},$$

where

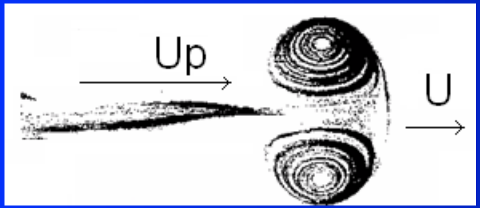
$$\alpha = \frac{E}{\rho I \Gamma^3},$$

$$B = U\sqrt{\frac{\pi I}{\rho \Gamma^3}},$$

$$b = R_0\sqrt{\frac{\rho \pi \Gamma}{2I}},$$

$$N = \frac{U}{U_p} = \frac{UI}{2E}.$$

Criterion for the pinch-off



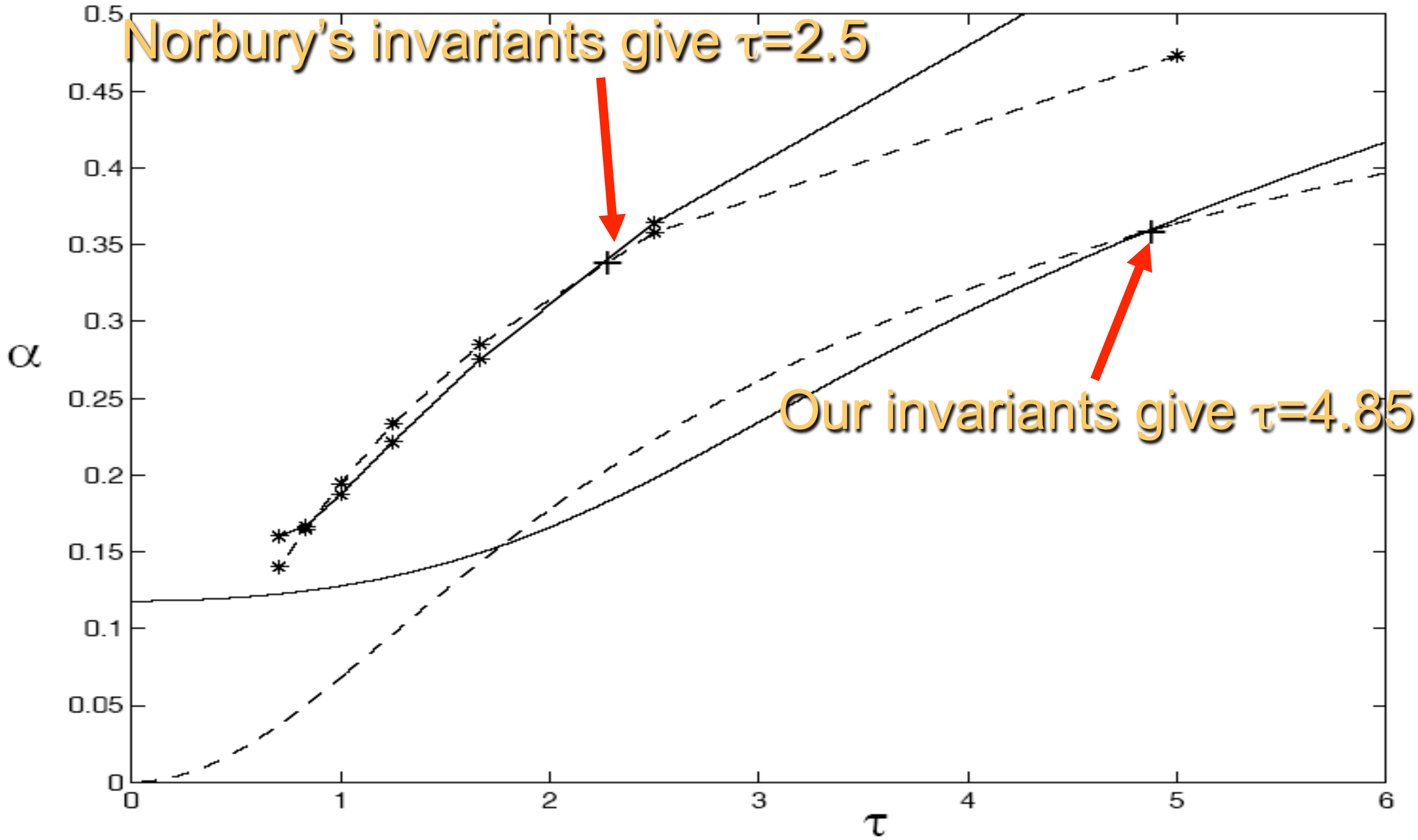
$$U = \frac{D^2}{4R_0^{\,2}} U_p$$

Results

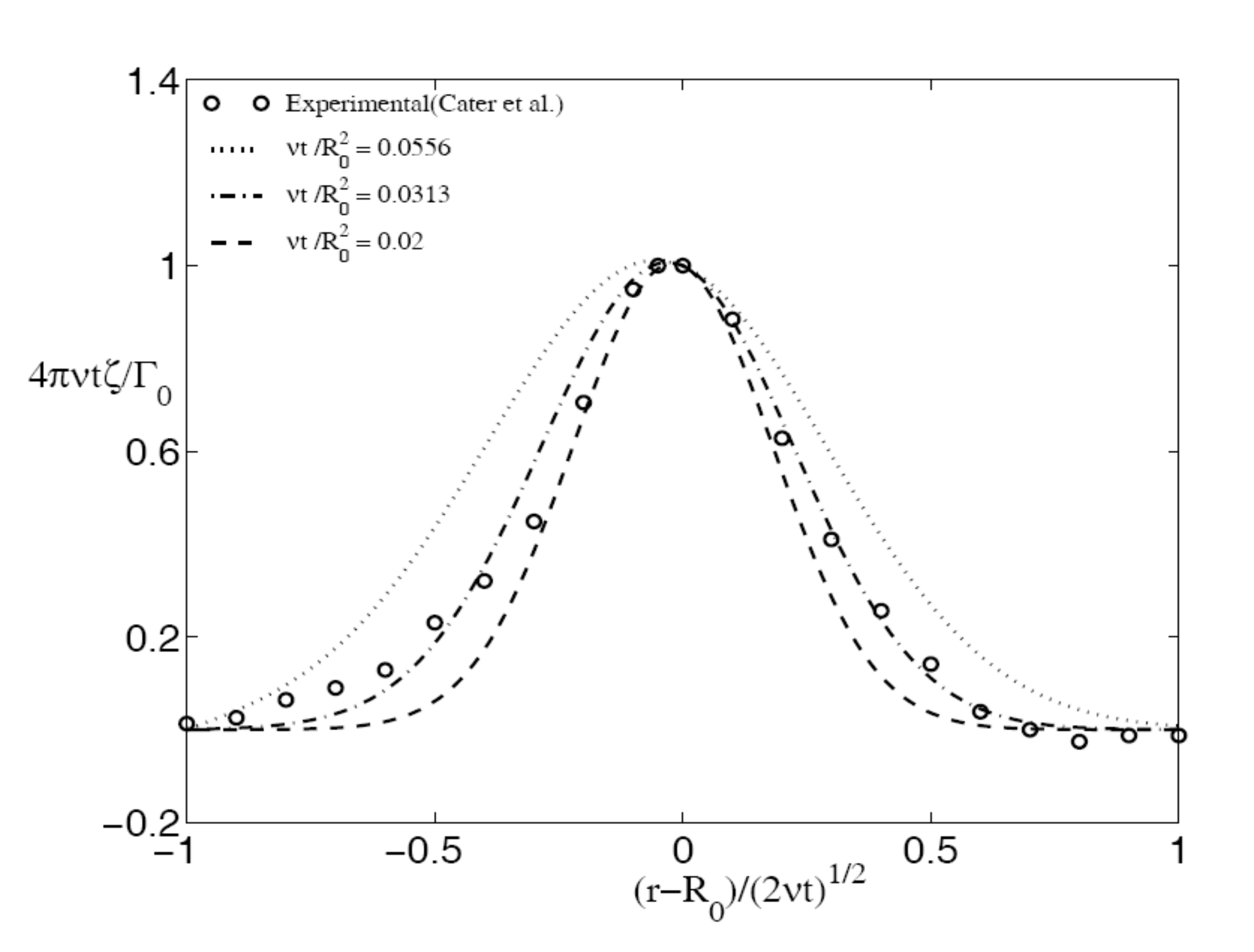
$$\frac{L}{D} = \frac{\pi\sqrt{2}}{4b^2B} = 3.5$$

B=0.6907,b =0.6775, Norbury's data

B=0.6350,b =0.7071, our data



Comparison of the results with experimental data (Cater et al., 2004) for Re=2000.



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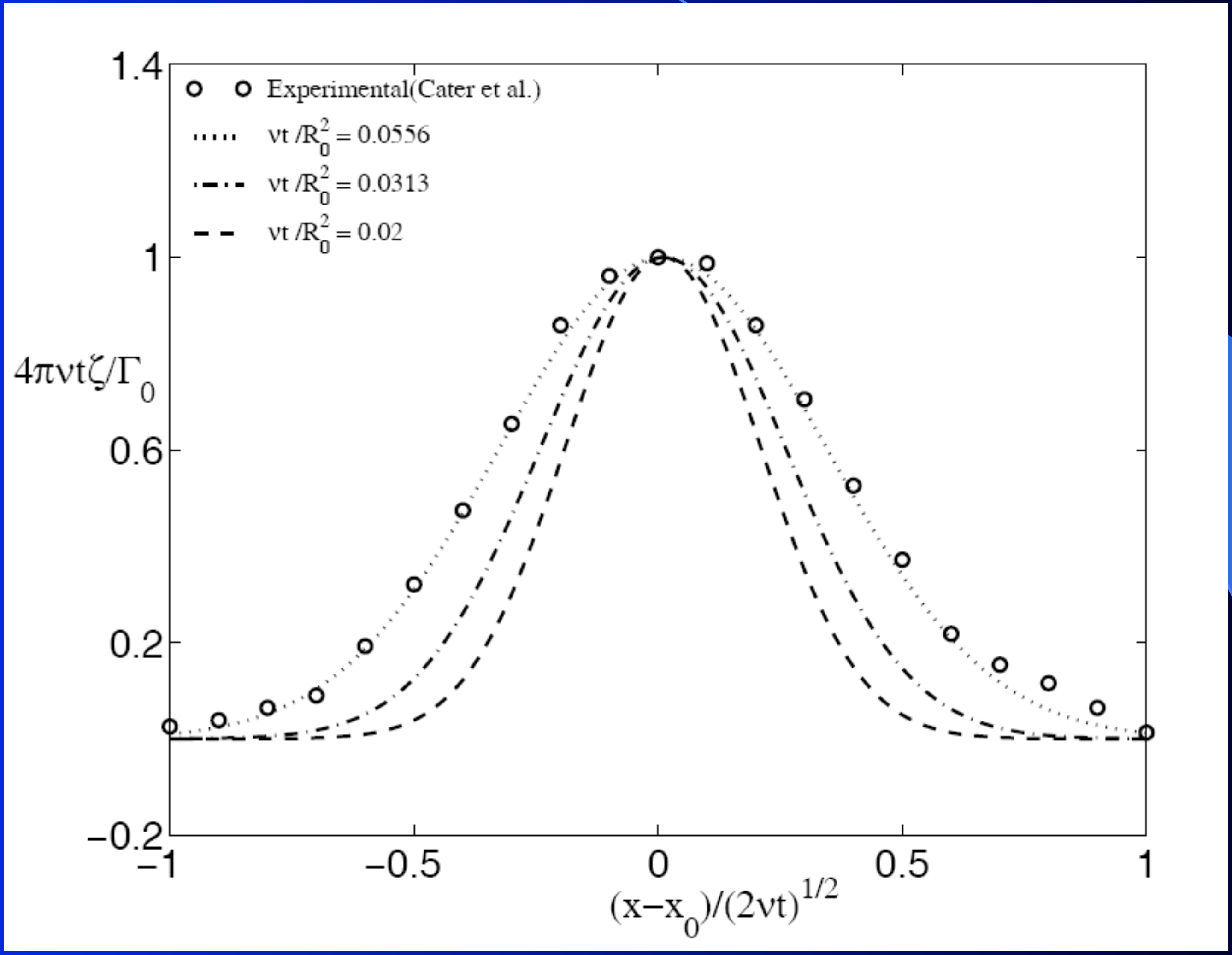

Fukumoto&Kaplanski, 2007, submitted for publication.

Conclusions

- A new vortex ring model, valid in the entire range of times, has been suggested. The model agrees with earlier reported models for the initial and decaying stages of vortex ring development and experimental results. It can be considered as a viscous analog to the Norbury family of rings.
- The model is shown to be useful for modelling high-Reynolds-number ring flows and turbulent vortex rings. It predicts the ‘formation number’ L/D for ‘optimal’ vortex rings.

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Thank for your attention.