Effect on stability of separating a velocity jump from a density jump

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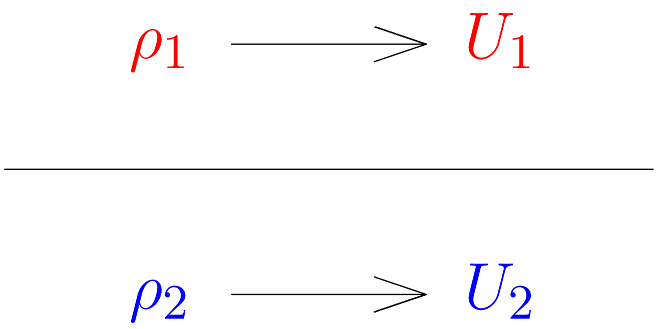
The basic flow

Classic two-layered stratified Kelvin-Helmholtz flow:



Coincident velocity and density interfaces.

The basic flow

Classic two-layered stratified Kelvin-Helmholtz flow:

$$\rho_1 \longrightarrow U_1$$

$$\rho_2 \longrightarrow U_2$$

Coincident velocity and density interfaces.

Instability mechanism in jets (and wakes).

The basic flow

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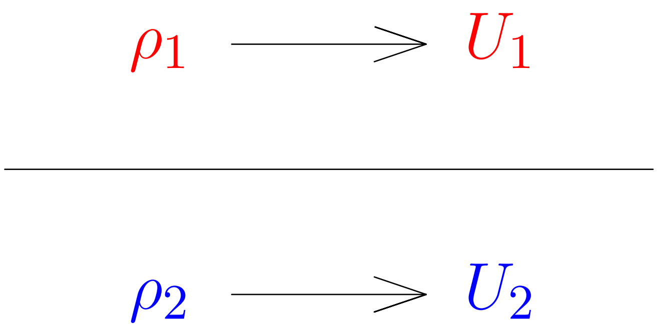
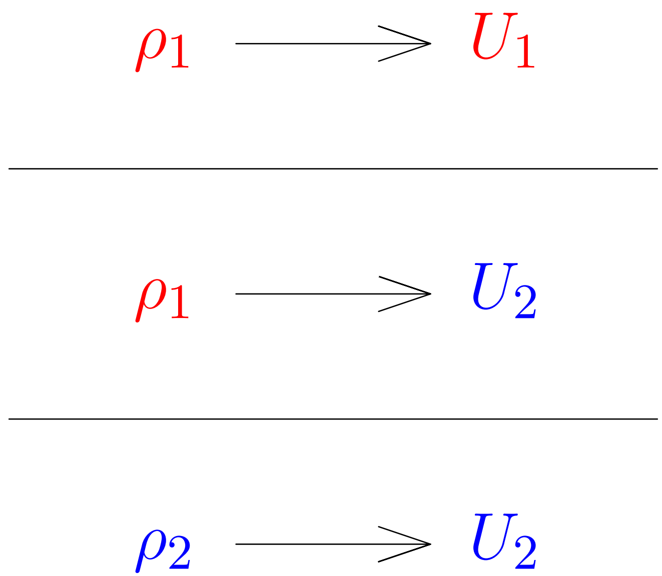


Coincident velocity and density interfaces.

Instability mechanism in jets (and wakes).

A three-layered model:



Distinct velocity and density interfaces.

Linearized waves in homogeneous fluid

- Add a small disturbance to a parallel shear layer:

$$\begin{aligned}\tilde{u} &= U(y) + \epsilon u(y) \exp \mathrm{i}(\alpha x - \omega t) \\ \tilde{v} &= \epsilon v(y) \exp \mathrm{i}(\alpha x - \omega t) \\ \tilde{p} &= \epsilon p(y) \exp \mathrm{i}(\alpha x - \omega t)\end{aligned}$$

  where $\epsilon \ll 1$.
- Substitute into the Navier–Stokes equations.
- Neglect $O(\epsilon^2)$ terms (linearize), and viscosity.
- Eliminate u and p to give the Rayleigh equation:

$$(U - c)(v'' - \alpha^2 v) - U'' v = 0$$

  where $c = \omega/\alpha$ and $v = 0$ on boundaries.

Dispersion relation

- In a region were $U =$ const., the Rayleigh equation reduces to

$$v'' - \alpha^2 v = 0$$

with solutions

$$v = A\exp(\alpha y) + B\exp(-\alpha y).$$

- Boundary conditions at an interface:
  - Interface moves with fluid (kinematic condition).
  - Pressure is continuous (dynamic condition).
- Eliminating the arbitrary constants gives the dispersion relation:

$$\Delta(\alpha, \omega) = 0.$$

Temporal instability

- Any initial condition can be expressed as a superposition of normal modes with real α.
- Each normal mode evolves with an ω satisfying the dispersion relation.
- The solution for ω could be complex:

$$\omega = \omega_r + \mathrm{i}\omega_i.$$

- If there exists a real α with $\omega_i > 0$, then there is growth in time:

$$\exp \mathrm{i}(\alpha x - \omega t) = [\exp(\omega_i t)][\exp \mathrm{i}(\alpha x - \omega_r t)].$$

Stratified Kelvin-Helmholtz instability

- We shall only consider stable stratification: $\rho_2 \geq \rho_1$.
- Special cases:
  - Internal gravity waves: $U_1 = U_2$,

$$c = U_2 \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)}}.$$

  - Homogeneous Kelvin-Helmholtz instability: $\rho_1 = \rho_2$,

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{\mathrm{i}}{2}(U_1 - U_2).$$

Stratified Kelvin-Helmholtz instability

- General case:

$$c = \frac{(\rho_1 U_1 + \rho_2 U_2)}{(\rho_1 + \rho_2)} \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)} - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2}(U_1 - U_2)^2}.$$

- Unstable short waves: stable long waves:

$$c \sim \bar{U} \pm \mathrm{i}\frac{\sqrt{\rho_1 \rho_2}}{(\rho_1 + \rho_2)}(U_1 - U_2), \qquad c \sim \bar{U} \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)}}.$$

where

$$\bar{U} = \frac{(\rho_1 U_1 + \rho_2 U_2)}{(\rho_1 + \rho_2)}.$$

- Instability for short enough waves: $\alpha > \alpha_c = \dfrac{(\rho_2^2 - \rho_1^2)g}{\rho_1 \rho_2 (U_1 - U_2)^2}$

Three-layered model

- Dispersion relation is fourth order in c.
- Nondimensionalize so that limiting cases can be examined:

  lengths by h, velocities by $(U_1 + U_2)/2$, density by ρ_1.

- Introduce dimensionless parameters:

$$r = \frac{(U_1 - U_2)}{(U_1 + U_2)}, \quad b = \frac{4gh(\rho_2 - \rho_1)}{(U_1 + U_2)^2(\rho_1 + \rho_2)} = F^{-2},$$

$$\rho = \frac{\rho_2}{\rho_1} \geq 1.$$

Three-layered model

- Short waves, $\alpha \gg 1$, (thick middle layer):

$$c \sim 1 \pm \mathrm{i}r, \quad 1 - r \pm \sqrt{\frac{b}{\alpha}}$$

or in dimensional form:

$$c \sim \frac{1}{2}(U_1 + U_2) \pm \frac{\mathrm{i}}{2}(U_1 - U_2), \quad U_2 \pm \sqrt{\frac{g(\rho_2 - \rho_1)}{\alpha(\rho_1 + \rho_2)}}.$$

- Homogeneous Kelvin-Holmholtz instability on the velocity interface.
- Internal gravity waves on the density interface.
- (As expected).

Three-layered model

- Long waves, $\alpha \ll 1$, (thin middle layer):

$$c \sim 1 - r\frac{(\rho-1)}{(\rho+1)} \pm \sqrt{\frac{b}{\alpha}}, \quad 1 - r \pm 2\mathrm{i}r\sqrt{\alpha},$$

  or in dimensional form:

$$c \sim \bar{U} \pm \sqrt{\frac{g(\rho_2-\rho_1)}{\alpha(\rho_1+\rho_2)}}, \quad U_2 \pm \mathrm{i}(U_1-U_2)\sqrt{h\alpha}.$$

- Internal gravity waves, like the two-layered case.
- Instability, NOT like the two-layered case!

Three-layered model

- Long waves, $\alpha \ll 1$, in zero buoyancy limit, $b = 0$:

$$c \sim 1 - r\frac{(\rho - 1)}{(\rho + 1)} \pm 2\mathrm{i}\frac{\sqrt{\rho}}{(\rho + 1)}r, \quad c = 1 - r, \quad 1 - r,$$

  or in dimensional form:

$$c \sim \bar{U} \pm \mathrm{i}\frac{\sqrt{\rho_1 \rho_2}}{(\rho_1 + \rho_2)}(U_1 - U_2), \quad c = U_2, \quad U_2,$$

- as in short-wave limit of two-layered case.

Three-layered model

- Long waves, $\alpha \ll 1$, and small buoyancy, $b = b_0\alpha$:

$$c \sim 1 - r\frac{(\rho-1)}{(\rho+1)} \pm \sqrt{b_0 - \frac{4\rho r^2}{(1+\rho)^2}}, \quad 1 - r \pm 2\sqrt{\frac{b_0(\rho+1)r^2\alpha}{4r^2 - b_0(\rho+1)}}$$

  or in dimensional form:

$$c \sim \bar{U} \pm \sqrt{\frac{g(\rho_2-\rho_1)}{\alpha(\rho_1+\rho_2)} - \frac{\rho_1\rho_2}{(\rho_1+\rho_2)^2}(U_1-U_2)^2},$$

$$c \sim U_2 \pm (U_1-U_2)\sqrt{\frac{\alpha h g(\rho_2-\rho_1)}{\alpha\rho_1(U_1-U_2)^2 - g(\rho_2-\rho_1)}}.$$

- New mode is unstable for strong enough stable stratification!

Three-layered model

- For long waves, $\alpha \ll 1$, and small buoyancy, $b = b_0\alpha$:
  - The K-H mode is unstable for short enough waves:
$$\alpha > \frac{g(\rho_2^2 - \rho_1^2)}{\rho_1\rho_2(U_1 - U_2)^2}.$$
  - The new mode is unstable for long enough waves:
$$\alpha < \frac{g(\rho_2 - \rho_1)}{\rho_1(U_1 - U_2)^2}.$$
  - Waves are stable for
$$1 < \frac{\rho_1(U_1 - U_2)^2}{g(\rho_2 - \rho_1)}\alpha < 1 + \frac{\rho_1}{\rho_2}$$
- But for $b = O(1)$, this stable interval closes up, e.g. at $r = 1$ and $\rho = 2$, all waves are unstable for $b > 0.0073$.

Smooth profiles

- The generalization of the Rayleigh equation for inviscid disturbances to a stratified flow is the Taylor-Goldstein equation:

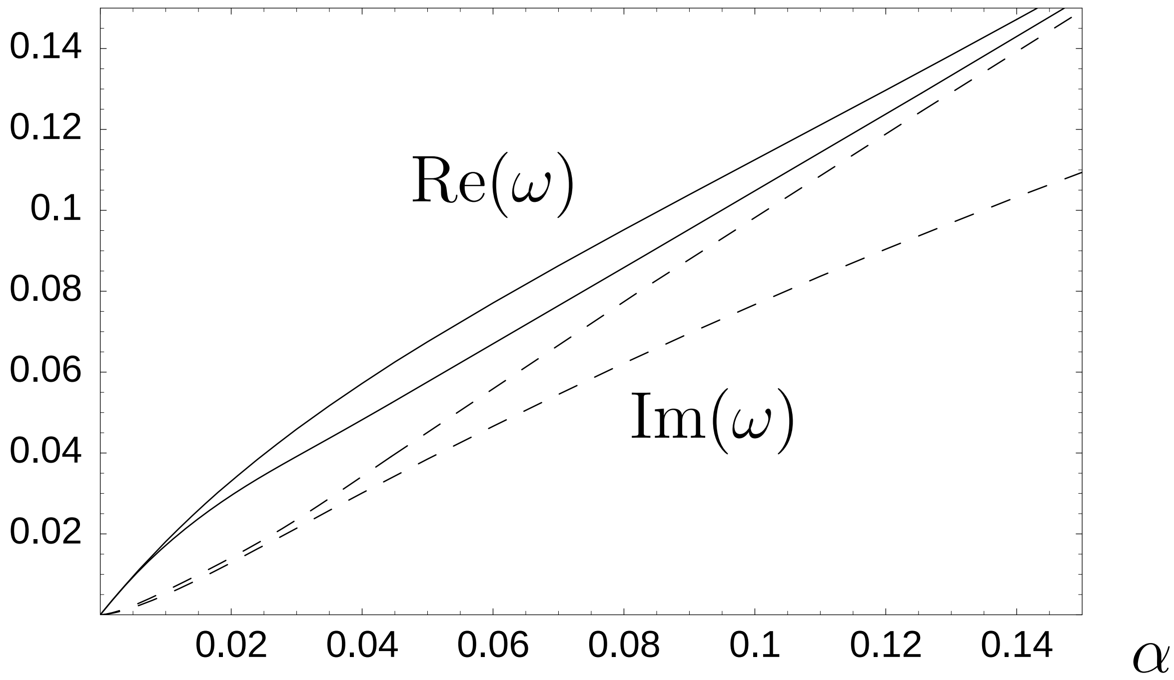
$$(U-c)(v''-\alpha^2 v) - U''v - \frac{b\rho_B'}{\rho_B(U-c)}v + \frac{\rho_B'}{\rho_B}[(U-c)v' - U'v] = 0$$

  (including density variation in the inertia terms).
- Consider basic velocity U and basic density ρ_B:

$$\begin{aligned} U &= 1 + r\tanh y, \quad r = \frac{U_1 - U_2}{U_1 + U_2} \\ \rho_B &= 1 + \delta\tanh(y+h), \quad \delta = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}. \end{aligned}$$

Numerical Taylor-Goldstein solutions

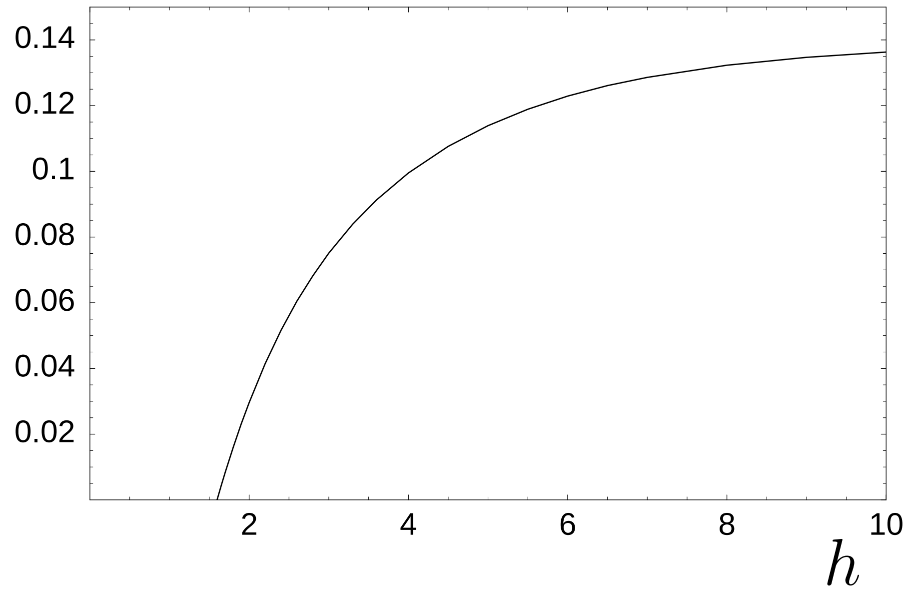
- Comparison between numerical Taylor-Goldstein solutions and analytic results of 3-layered model for $r = -1$, $b = 1$, $\rho_2/\rho_1 = 2$:



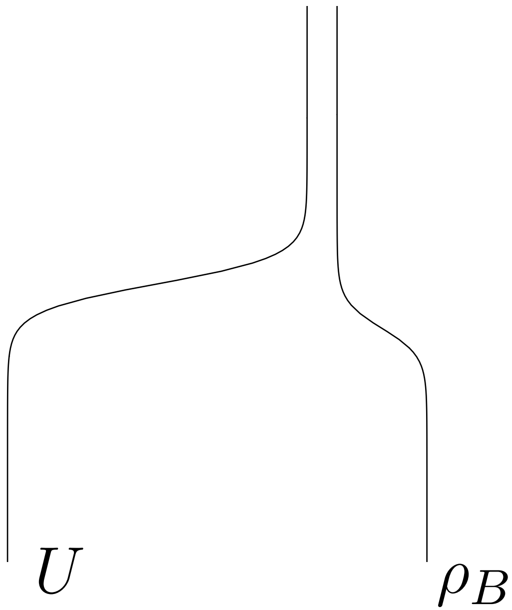
- Independent confirmation of long-wave instability.

Merging density and velocity layers

$\mathrm{Im}(\omega)$



$\alpha = 0.2$, $r = -1$, $b = 1$, $\rho_2/\rho_1 = 2$



Layer separation
$h = 1.6$

Conclusions

- If the velocity jump and density jump do not coincide, then there is qualitatively different behaviour to the K-H case where they do coincide.
- Stably stratified K-H flow is stable for long waves.
- But three-layered flow is unstable for long waves.
- The new mode is destabilized by increasing stable stratification, and stabilized by increasing shear.
- Could be important, e.g., in wave generation when there is a shear layer in the air above a body of water.
- Results have been confirmed by numerical solution for smooth profiles.