Vortex ring models

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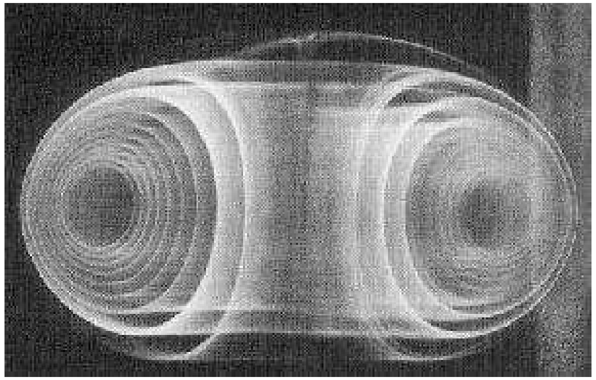
Vortex ring flow

Examples of vortex flows

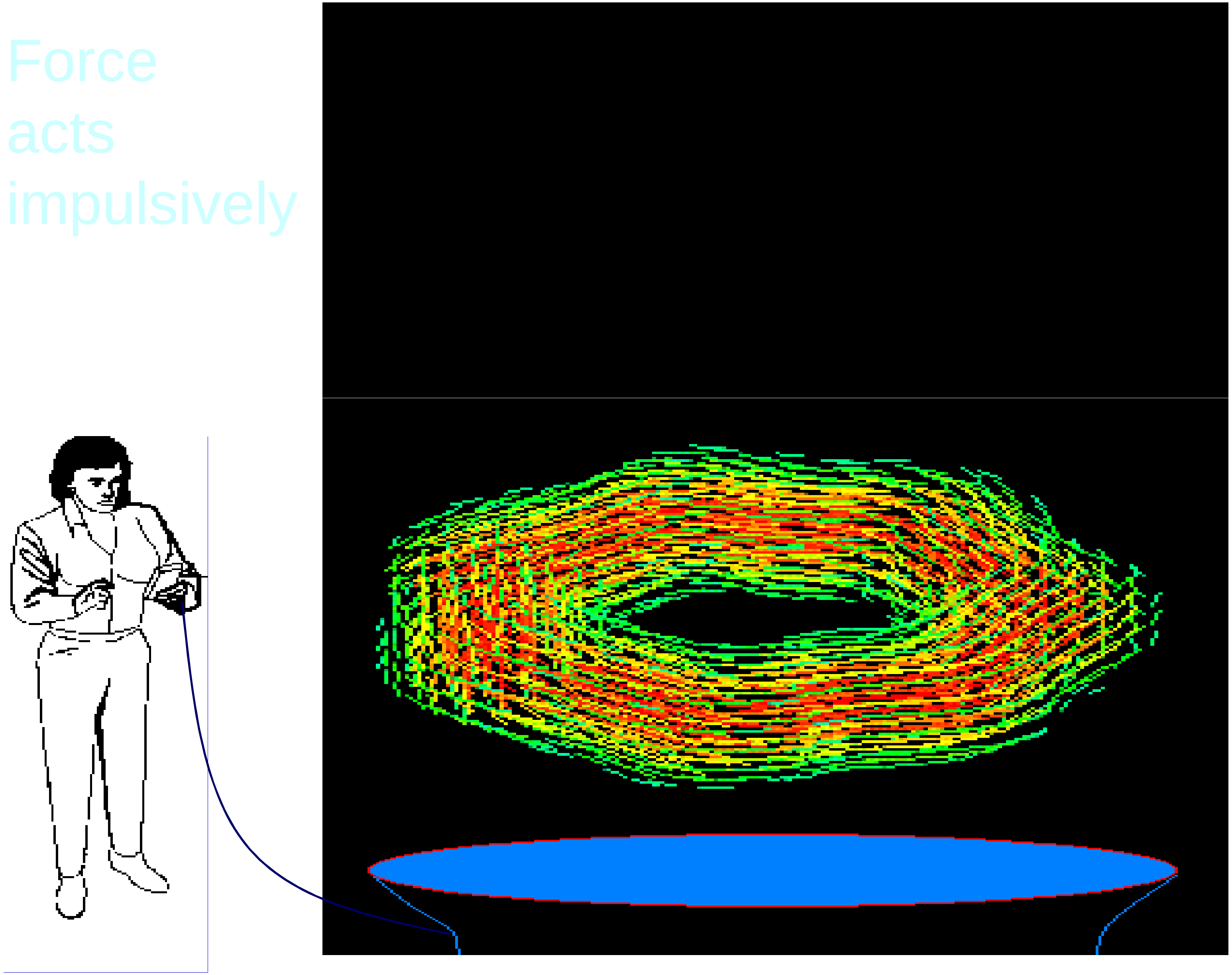


A VORTEX RING
At the right is a vortex ring generated by Professor T.T. Lim and his former colleagues at the University of Melbourne. The visualization technique appears to be by smoke.

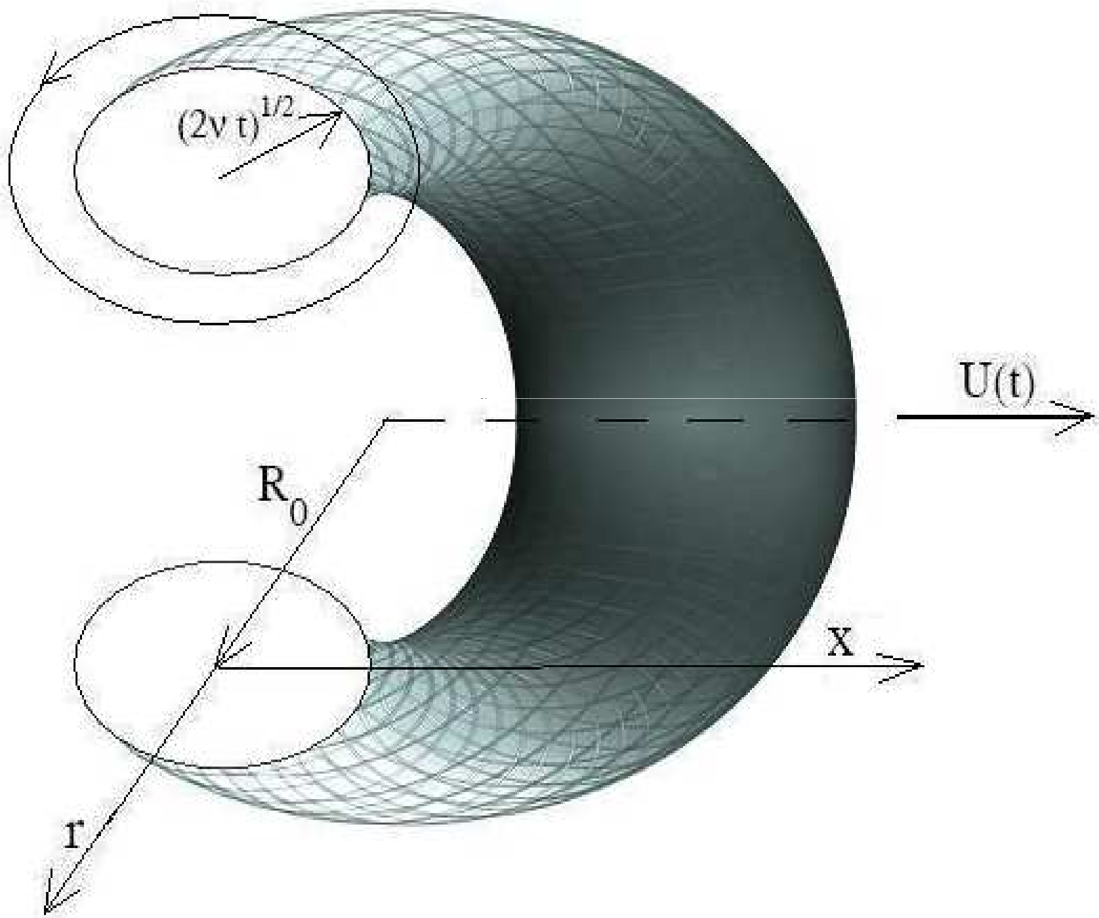
Virtual image of a vortex ring flow

www.applied-scientific.com/ MAIN/PROJECTS/NSF00/FAT_RING/Fat_Ring.html -

Force
acts
impulsively



Schematic view of a vortex ring



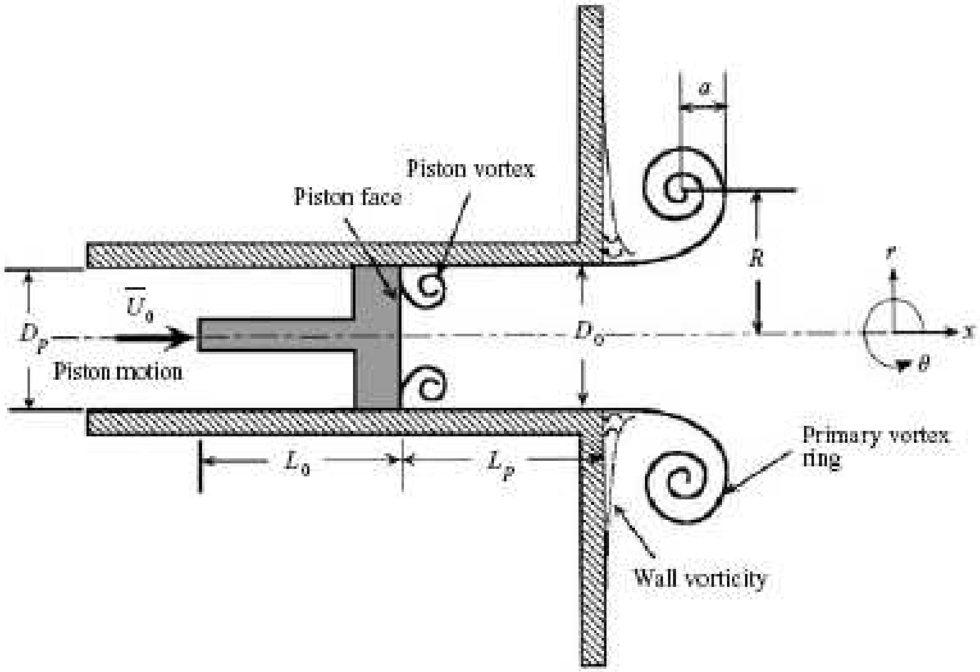


FIGURE 1. Geometric parameters of a piston/cylinder vortex ring generator. D_p is the piston diameter and D_0 is the orifice diameter. In this study $D_0 = D_p = 50$ mm. The mean piston velocity (or 'slug-flow velocity') is $\overline{U_0}$ and L_0 is the stroke length of the piston. L_p is the distance from the lip of the orifice to the piston face at the end of the piston stroke. For the vortex ring, R is the major radius of the toroidal vortex rings and a is an estimate of the radius of the core. Shaded areas indicate solid surfaces. Flow is from left to right.

Formulation of the problem

$$\frac{\partial \varsigma}{\partial t} + \frac{\partial}{\partial r}(v\varsigma) + \frac{\partial}{\partial x}(u\varsigma) = \nu\left[\frac{\partial^2 \varsigma}{\partial x^2} + \frac{\partial^2 \varsigma}{\partial r^2} + \frac{1}{r}\frac{\partial \varsigma}{\partial r} - \frac{\varsigma}{r^2}\right],$$

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \qquad v = -\frac{1}{r}\frac{\partial \Psi}{\partial x},$$

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = -r\varsigma.$$

$$\Psi(0,x) = \varsigma(0,x) = 0,$$

$$(x^2 + r^2)^{1/2} \to \infty : \Psi, \varsigma \to 0.$$

$$\varsigma(o,x,r) = \Gamma_0 \delta(x)\delta(r - R_0)$$

$$I = \pi\rho\int_0^\infty\int_{-\infty}^\infty r^2 \varsigma\, dx\, dr \qquad M = \frac{I}{\rho}$$

Basic solution

$$\sigma = \frac{r}{\ell}, \eta = \frac{x\text{-}x_0(t)}{\ell}, \quad \Phi = \frac{\Psi}{\varsigma_0 \ell^3}, \omega = \frac{\varsigma}{\varsigma_0},$$

$$\ell = \sqrt{2\nu t}, \varsigma_0 = A(M, \nu, R_0)\, t^{-3/2}$$

$$\tau = \frac{R_0}{\ell}$$

ring-to-core radius

$$\omega = exp\left(-\frac{1}{2}\left(\sigma^2 + \eta^2 + \tau^2\right)\right) I_1(\sigma\tau),$$

Translational velocity of the vortex ring as velocity of the vortex centroid:

$$U(t) = \frac{dx_0(t)}{dt}$$

$$x_0 = \frac{2\pi \int_0^{\infty} \int_{-\infty}^{\infty} r \; x \varsigma dx dr}{2\pi \int_0^{\infty} \int_{-\infty}^{\infty} r \varsigma dx dr}$$

3. VELOCITY OF THE VORTEX RING

A general expression for the velocity of a viscous vortex ring valid for arbitrary vorticity and stream function distributions was found in [8]

$$U = \int_{0}^{\infty}\int_{-\infty}^{\infty}(\Psi + 6xrv)\zeta \, dx\, dr \Big/ 2\int_{0}^{\infty}\int_{-\infty}^{\infty} r^2\zeta \, dx\, dr \qquad (3.1)$$

In [8] formula (3.1) was derived using the Lamb transformation [16] (the basis of this approach was first developed by Helmholtz [17]) for the velocity of a ring in an ideal fluid and the validity of its application to a viscous fluid was proved. Formula (3.1) gives the instantaneous velocity of a three-dimensional vortex centroid [9].

Using (3.1) and (1.7), we can find the velocity U in the form:

- Other way with the help of Fourier – Hankel transforms

(1)

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} - \frac{1}{z}\frac{\partial\psi}{\partial z} = -z f$$

$$\sigma = \frac{z}{L}, \quad \eta = \frac{x - x_0}{L}, \quad \Phi = \frac{\psi}{f_0 L^3}, \quad \omega = \frac{f}{f_0}$$

$$\frac{f_0 L^3}{L^2}\left[\frac{\partial^2\Phi}{\partial\sigma^2} + \frac{\partial^2\Phi}{\partial\eta^2} - \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}\right] = -\sigma L f_0 \omega$$

$$\frac{\partial^2\Phi}{\partial\sigma^2} + \frac{\partial^2\Phi}{\partial\eta^2} - \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma} = -\sigma\omega$$

$$\Phi = f(\sigma,\eta)\cdot\sigma$$

$$(f\sigma)'_\sigma = f'_\sigma \sigma + f$$

$$(f\sigma)''_\sigma = f''_\sigma \sigma + f'_\sigma + f'_\sigma$$

$$f''_\sigma \sigma + 2 f'_\sigma + f''_\eta \sigma - f'_\sigma - \frac{f}{\sigma} = -\sigma\omega$$

$$f''_\sigma \sigma^2 + f'_\sigma \sigma - f + f''_\eta \sigma^2 = -\sigma^2\omega$$

$$f''_\sigma + \frac{f'_\sigma}{\sigma} - \frac{f}{\sigma^2} + f''_\eta = -\omega \quad (1)$$

- Idea: To simplify solving of equations and integrating

Fourier transforms: good for Cartesian coordinate system

Fourier

$$\tilde{f}(\alpha,\mu) = \mathcal{F}\left[f(\sigma,\eta)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma,\eta)\, e^{-i\eta\alpha}\, d\eta$$

Hankel transforms: good for cylindrical coordinate system

Hankel

$$\tilde{f}(\mu,\alpha) = \mathcal{H}_1\left[f(\sigma,\varphi)\right] =$$

$$= \int_0^\infty f(\sigma,\varphi)\,\sigma\,J_1(\mu\sigma)\,d\sigma$$

Properties of the Hankel and Fourier transforms, respectively

and using properties for Hankel transforms

$$\mathcal{H}_1\left[f_{\sigma\sigma} + f_\sigma \frac{1}{\sigma} - f\frac{1}{\sigma^2}\right] = -\mu^2 \tilde{f}$$

and for Fourier transforms

$$\mathcal{F}\left[f_{zz}\right] = (i\alpha)^2 \mathcal{F}[f]$$

We find

$$-\mu^2 \tilde{f} - \alpha^2 \tilde{f} =$$

$$= -\mathcal{F}_1 \left(\mathcal{F} \left[\omega \right] \right) = \tilde{\omega}$$

$$\boxed{f_b'' + \frac{f_b'}{b} - \frac{f}{b^2} + f_z'' = -\omega} \quad (1) \rightarrow \tilde{f} = \frac{\tilde{\omega}}{\alpha^2 + \mu^2}$$

Fourier- Hankel integral transform

Hankel

$$\omega = exp\left(-\frac{1}{2}\left(\sigma^2\right)\right)(\sigma\tau)/2 \rightarrow \overline{\omega} = exp\left(-\frac{\mu^2}{2}\right)(\tau\mu)/2$$

Fourier

$$\omega = exp\left(-\frac{1}{2}\left(\eta^2\right)\right) \rightarrow \overline{\omega} = exp\left(-\frac{\alpha^2}{2}\right)$$

Join Fourier- Hankel integral transform with respect to σ and η

$$\omega = exp\left(-\frac{1}{2}\left(\sigma^2+\eta^2\right)\right)(\sigma\tau)/2 \rightarrow \bar{\omega} = exp\left(-\frac{\mu^2+\alpha^2}{2}\right)(\tau\mu)/2$$

$$f = \Phi / \sigma$$

$$\bar{f} = \frac{\bar{\omega}}{\alpha^2+\mu^2} \rightarrow \bar{f} = \frac{exp\left(-\frac{\mu^2+\alpha^2}{2}\right)}{\mu^2+\alpha^2}(\tau\mu)/2$$

U=

$$-6\int_0^\infty \sigma\left\{ f\omega\eta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f\frac{\partial}{\partial\eta}(\eta\omega)\,d\eta\right\}d\sigma = -6\int_0^\infty \sigma\left\{-\int_{-\infty}^{\infty} f\omega\,d\eta + \int_{-\infty}^{\infty} \eta^2 f\,\omega\,d\eta\right\}d\sigma$$

$$= 6E_1 - 6\int_0^\infty\int_{-\infty}^\infty \sigma\eta^2 f\omega\,d\eta\,d\sigma\,. \quad (22)$$

The result is

$$U = \frac{\pi}{M}\frac{\zeta_0^2 L^5}{2}\left[7E_1 - 6\int_0^\infty\int_{-\infty}^\infty \sigma\eta^2 f\omega\,d\eta\,d\sigma\right]. \quad (23)$$

The Fourier integral transform for $\eta^3\omega$ is the following

$$I_1(\sigma)\exp\left(-\frac{\tau^2+\sigma^2}{2}\right)\int_{-\infty}^{\infty}\eta^2\exp\left(-\frac{\eta^2}{2}\right)\exp(i\eta\alpha)\,d\eta = I_1(\sigma)\exp\left(-\frac{\tau^2+\sigma^2+\alpha^2}{2}\right)(1-\alpha^2).$$

(24)

Using the above results and the Parseval's theorem for Hankel's and Fourier transforms (also Plancherel theorem),

For Hankel transforms

$$\int_0^\infty f(r)g(r)r\ dr = \int_0^\infty F_\nu(k)G_\nu(k)k\ dk$$

For Fourier transforms

$$\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega,$$

$$\mathcal{F}^2 f(x) = f(-x), \quad \text{and} \quad \mathcal{F}^* = \mathcal{F}^{-1} = \mathcal{F}^3.$$

$$\eta^2\, exp(-\eta^2/2) \to (1-\alpha^2)\,exp(-\alpha^2/2) \to t^2\, exp(-t^2/2) = F^3$$

we can obtain using $\varsigma_0 = \dfrac{2M}{(4\pi\nu t)^{3/2} R_0} = \dfrac{2M\tau^3}{(2\pi)^{3/2} R_0^4}$

$$U = \frac{M\tau}{4\pi^2 R_0^3} \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\mu exp\left(-\mu^2-\alpha^2\right)(\tau\mu)^2}{4(\mu^2+\alpha^2)}\left(1+6\alpha^2\right) d\mu d\alpha$$

Using (13), (14) and the Parseval's theorem for Hankel's transforms, we can obtain

$$U = \frac{M\tau}{4\pi^2 R_0^3} \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\mu \exp(-\mu^2 - \alpha^2) J_1^2(\tau\mu)}{\mu^2 + \alpha^2} (1 + 6\alpha^2) d\mu d\alpha . \qquad (25)$$

Integration with respect to α gives the final expression for U in the form:

$$U = \frac{M\tau}{4\pi^2 R_0^3} \int_0^{\infty} \{\pi(1 - \operatorname{erf}(\mu))(1 - 6\mu^2) + 6(\pi)^{1/2} \mu \exp(-\mu^2)\} J_1^2(\tau\mu) d\mu . \qquad (26)$$

The integral (26) can be expressed in the finite form

$$U = \frac{M\tau}{4\pi^2 R_0^3} \left\{ 3(\pi)^{1/2} \exp\left(-\frac{\tau^2}{2}\right) I_1\left(\frac{\tau^2}{2}\right) + \right.$$

$$+ \frac{1}{12} (\pi)^{1/2} \tau^2 \, {}_2F_2\left[\left\{\frac{3}{2}, \frac{3}{2}\right\}, \left\{\frac{5}{2}, 3\right\}, -\tau^2\right]$$

$$\left. - \frac{3(\pi)^{1/2}}{5} \tau^2 \, {}_2F_2\left[\left\{\frac{3}{2}, \frac{5}{2}\right\}, \left\{2, \frac{7}{2}\right\}, -\tau^2\right] \right\}, \qquad (27)$$

where ${}_2F_2$ is the generalized hypergeometric function [14].

The appropriate formula for the kinetic energy can be obtained in a similar manner. We start from the definition

$$E = \pi\rho \int_0^{\infty} \int_{-\infty}^{\infty} \zeta \Psi \, dx \, dr = \zeta_0^2 \ell^5 \pi\rho \int_{-\infty}^{\infty} \int_0^{\infty} \sigma f \, \omega \, d\sigma \, d\eta. \tag{10}$$

$$E = \frac{M^2 \rho \tau}{2\pi^2 R_0^3} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\mu \exp(-\mu^2 - \alpha^2) J_1^2(\tau\mu)}{\mu^2 + \alpha^2} d\mu \, d\alpha. \tag{13}$$

Hence, after integration with respect to α, the kinetic energy can be expressed in closed form as

$$E = \frac{M^2 \rho \tau}{2\pi^2 R_0^3} \int_0^{\infty} \{\pi[1 - \operatorname{erf}(\mu)]\} J_1^2(\tau\mu) \, d\mu$$

$$= \frac{M^2 \rho \tau}{2\pi^2 R_0^3} \frac{1}{12} (\pi)^{1/2} \tau^2 {}_2F_2\left(\left\{ \frac{3}{2}, \frac{3}{2} \right\}, \left\{ \frac{5}{2}, 3 \right\}, -\tau^2 \right),$$

where ${}_2F_2$ is the generalized hypergeometric function.[18]

Translational velocity of the vortex ring

$$U = \frac{\Gamma_0 \theta^{3/2}}{96\sqrt{2\pi}R_0}\left\{\frac{72}{\theta}\exp\left(-\frac{\theta}{4}\right) I_1(\frac{\theta}{4}) + {}_2F_2\left[\left\{\frac{3}{2},\frac{3}{2}\right\},\left\{\frac{5}{2},3\right\},-\frac{\theta}{2}\right] - \frac{36}{5}\,{}_2F_2\left[\left\{\frac{3}{2},\frac{5}{2}\right\},\left\{2,\frac{7}{2}\right\},-\frac{\theta}{2}\right]\right\}$$

where $\theta = R_0^2/(\nu t)$, R_0 , and Γ_0 are initial ring radius and circulation,
ν -is kinematic viscosity, I_1 is the modified Bessel function.

$$_2F_2[a_1, a_2; b_1, b_2; x] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k x^k}{(b_1)_k (b_2)_k k!},$$

$$(\alpha)_0 = 1;\ \ (\alpha)_1 = \alpha;\ \ (\alpha)_k = \alpha(\alpha+1)\ldots\ldots\ldots(\alpha+k-1)\ \ (k \ge 2)$$

Invariants: circulation and vorticity impulse

$$\Gamma = \int_{-\infty}^{\infty}\int_{0}^{\infty} \varsigma dxdr = \frac{M}{\pi R_0^2}\left(1 - exp\left(-\frac{\theta}{4}\right)\right)$$

$$I = \pi\rho \int_{0}^{\infty}\int_{-\infty}^{\infty} r^2 \varsigma dxdr = const.$$

Limits at small and large times.
Expansions in series of U by $\theta = R_0^2/(\nu t)$:

$\nu t \ll R_0^2$

$$U_s = \frac{\Gamma_0}{4\pi R_0}\left[\log(4\sqrt{\theta}) - \frac{1}{2}(\log 2 - \gamma + 1) - \frac{9}{2}(\log(4\sqrt{\theta}) - \frac{1}{2}(\log 2 - \gamma + 2))\frac{1}{\theta} + \frac{105}{32}\frac{1}{\theta^2} + O(\frac{1}{\theta^3})\right]$$

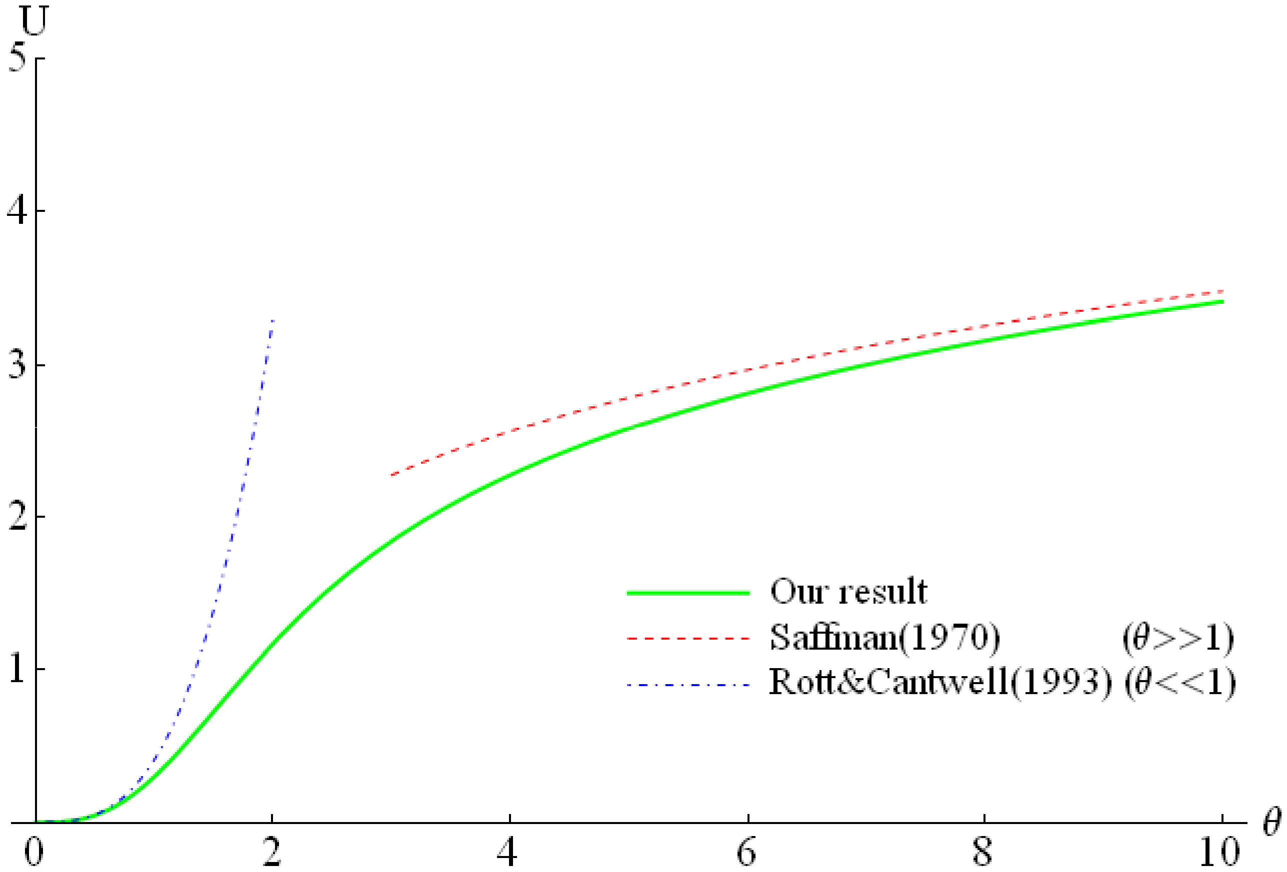
Saffman, 1970

$\nu t \gg R_0^2$

$$U_l = \frac{7I\theta^{3/2}}{240 R_0^3\sqrt{2}\pi^{3/2}}\left[1 - \frac{33}{196}\theta + \frac{125}{6272}\theta^2 + O(\theta^3)\right]$$

Rott & Cantwell, 1994

Our result unites results by Saffman and Rott&Cantwell



Kinetic energy of the vortex ring

$$E = \frac{\sqrt{\pi}\Gamma_0^2 \theta^{3/2} R_0}{48\sqrt{2}} \; {}_2F_2\left(\left\{\frac{3}{2},\frac{3}{2}\right\},\left\{\frac{5}{2},3\right\},-\frac{\theta}{2}\right)$$

Limits at small and large times. Expansions in series of E by $\theta = R_0^2/(\nu t)$:

$\nu t << R_0^2$

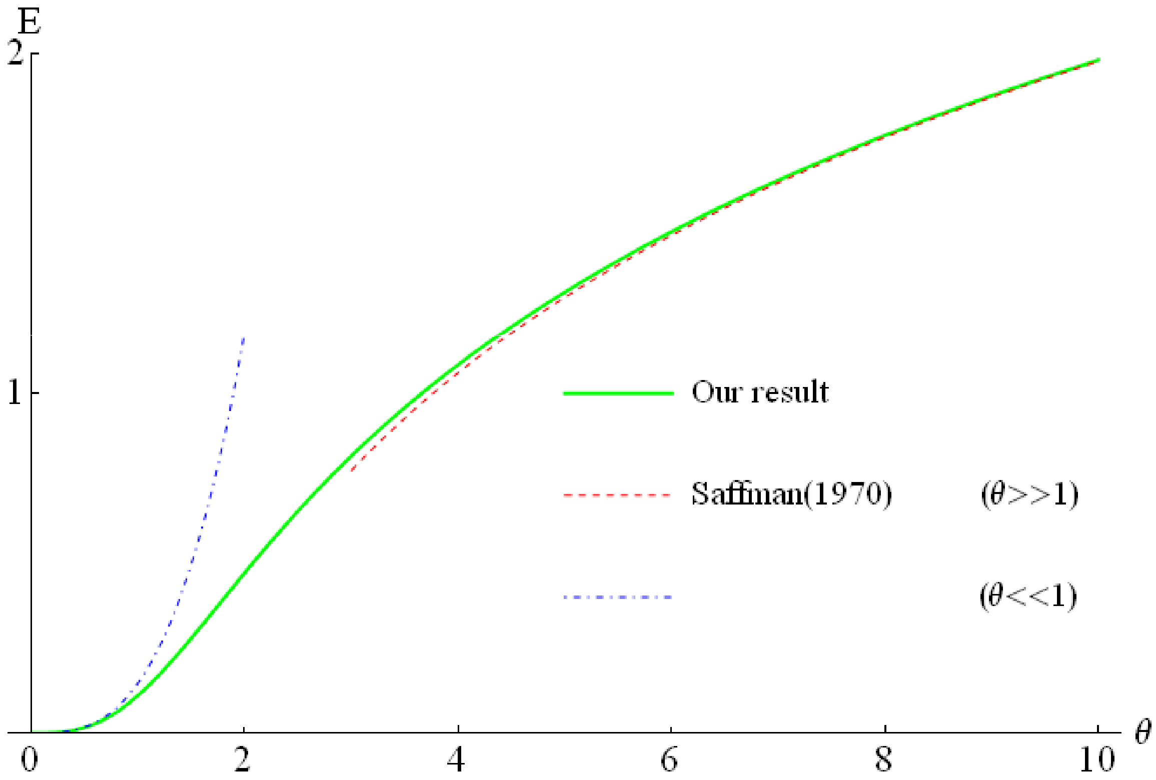
$$E_s = \frac{\Gamma_0^2 R_0}{2}\left[\log(4\sqrt{\theta}) - \frac{1}{2}(\log 2 - \gamma + 4) + \frac{3}{4}\frac{1}{\theta} + \frac{15}{32}\frac{1}{\theta^2} + O(\frac{1}{\theta^3})\right]$$

Saffman, 1970

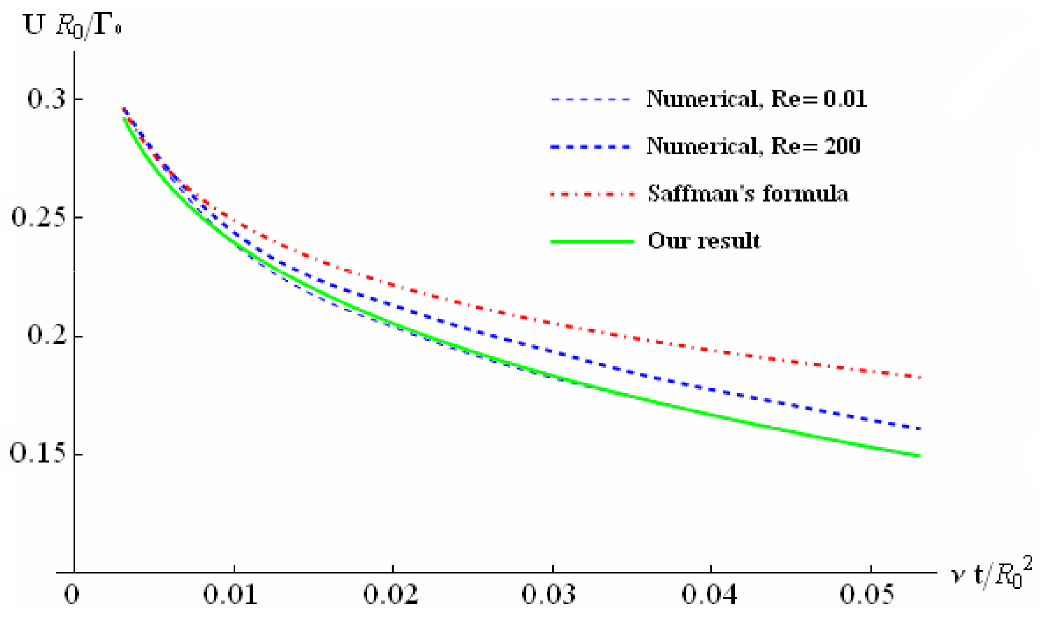
$\nu t >> R_0^2$

$$E_l = \frac{\Gamma^2 \theta^{3/2}}{48\sqrt{2} R_0^3}\left[1 - \frac{3}{20}\theta + \frac{15}{896}\theta^2 + O(\theta^3)\right]$$

Energy: comparison with Saffman's result



Velocity: comparison with numerical study by Stanaway *et al*. (1988)

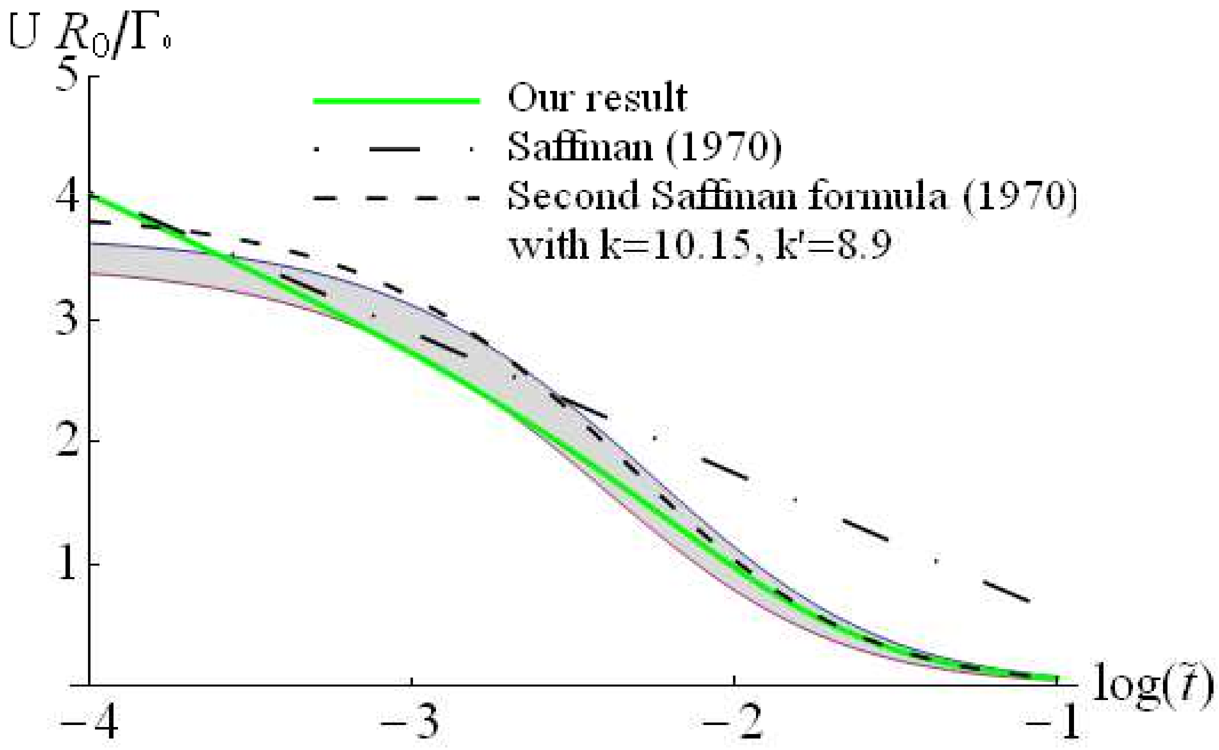


Second Saffman's formula for $\nu t \approx R_0^2$

$$U = \frac{\Gamma_0}{R_0} \frac{16\pi}{k} (1 + 16 k' t^*)^{-3/2},$$

where $t^* = \nu t / (4 D_0^2)$, k and k´ are tunable constants .

Comparison with second Saffman formula and experimental data by Weingand&Gharib (1997) for 830<<Re<<1650. Experimental data lie between curves with k =14.4; k′=7.8 and k=14.5; k′=7.5, respectively.



Numerical simulation

$$\frac{\partial \varsigma^*}{\partial t^*} + \frac{\partial}{\partial r^*}\left(-\frac{1}{r^*}\frac{\partial \Psi^*}{\partial x^*}\varsigma^*\right) + \frac{\partial}{\partial x^*}\left(\left(\frac{1}{r^*}\frac{\partial \Psi^*}{\partial r^*}\right)\varsigma^*\right) = \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 \varsigma^*}{\partial x^{*2}} + \frac{\partial^2 \varsigma^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial \varsigma^*}{\partial r^*} - \frac{\varsigma^*}{r^{*2}}\right]$$

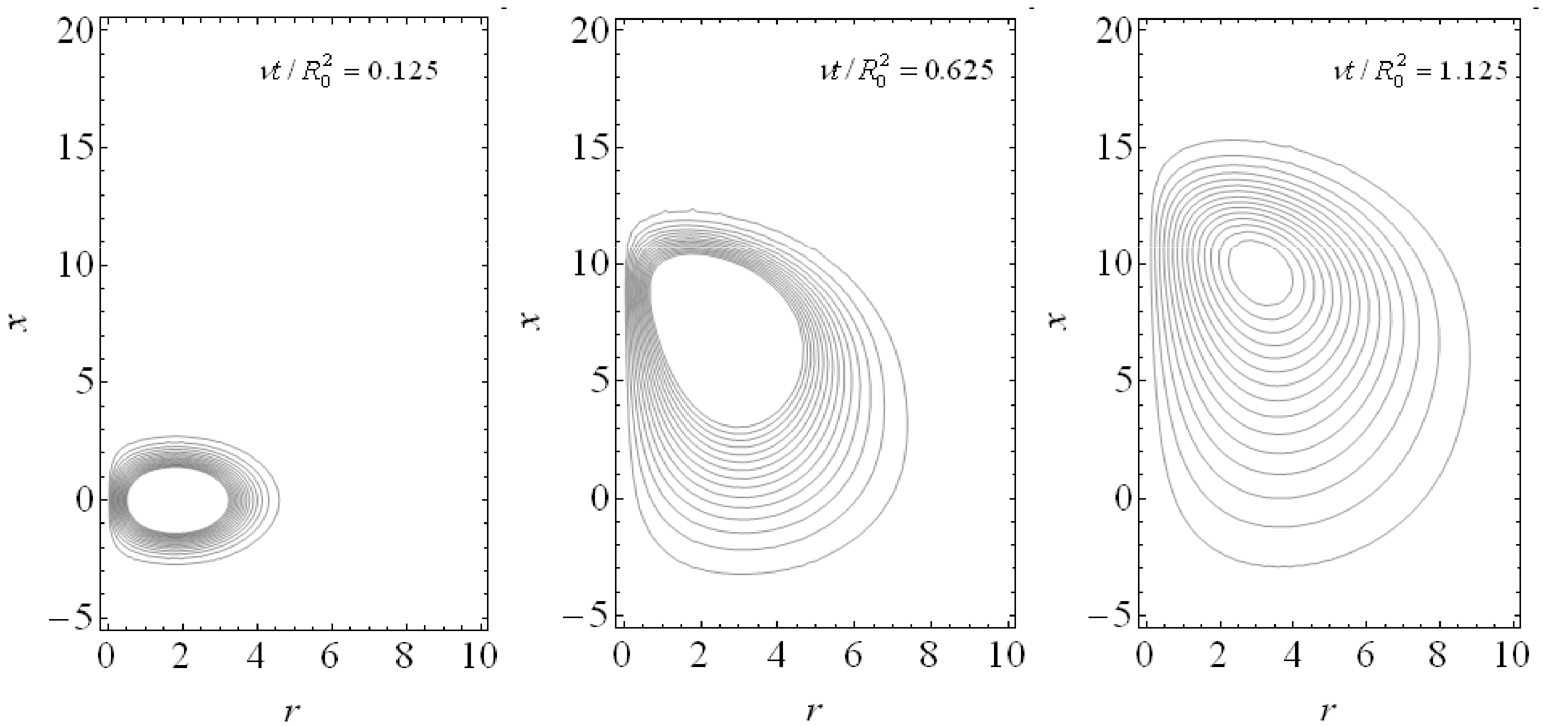
$$\frac{\partial^2 \Psi^*}{\partial r^{*2}} + \frac{\partial^2 \Psi^*}{\partial x^{*2}} - \frac{1}{r^*}\frac{\partial \Psi^*}{\partial r^*} = -r^*\varsigma^*$$

$$t^* = \frac{t\,\Gamma_1}{R_1^2},\ x^* = \frac{x}{R_1}, r^* = \frac{r}{R_1},\ \varsigma^* = \frac{\varsigma R_1^2}{\Gamma_1}, \Psi^* = \frac{\Psi}{\Gamma_1 R_1},\ \mathrm{Re} = \frac{\Gamma_1}{\nu}$$

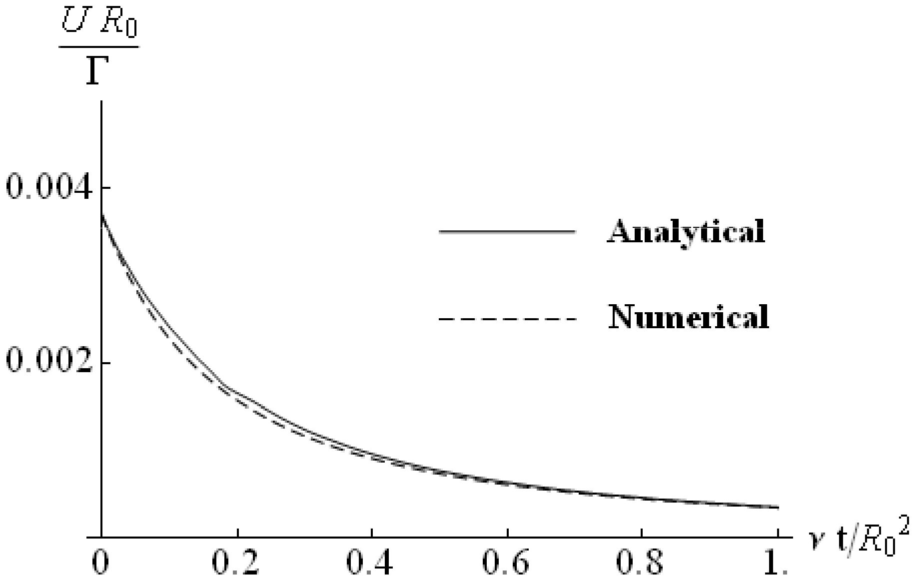
$$\bar{t} = \bar{t}_0 + \frac{\nu t}{R_0^2} = \frac{1}{2\theta_0^2} + \frac{t^*}{\theta_0^2\,\mathrm{Re}}$$

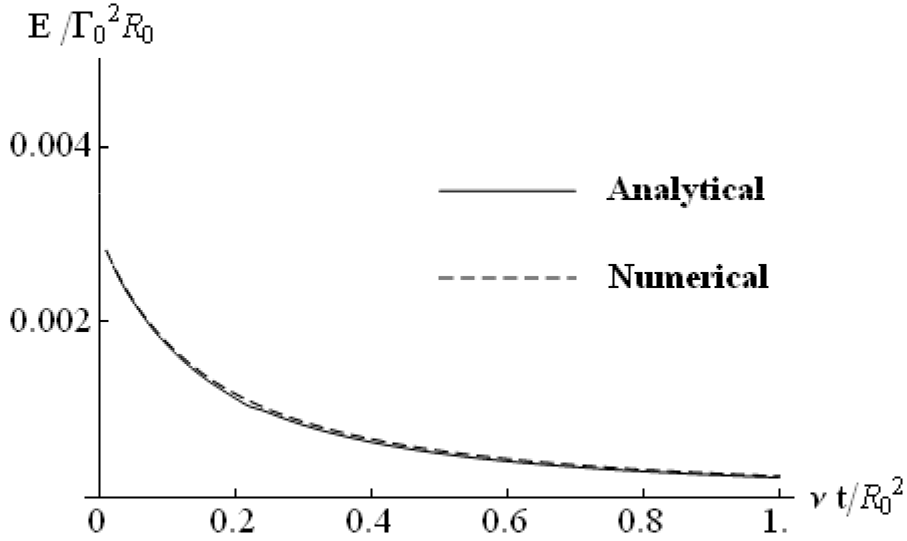
$$\varsigma^* = \varsigma_0^* \exp\left(-\frac{r^{*2} + x^{*2} + \theta_0^2}{2}\right) I_1\left(r^*\theta_0\right)$$

Re=1500



Comparison of analytical and numerical results

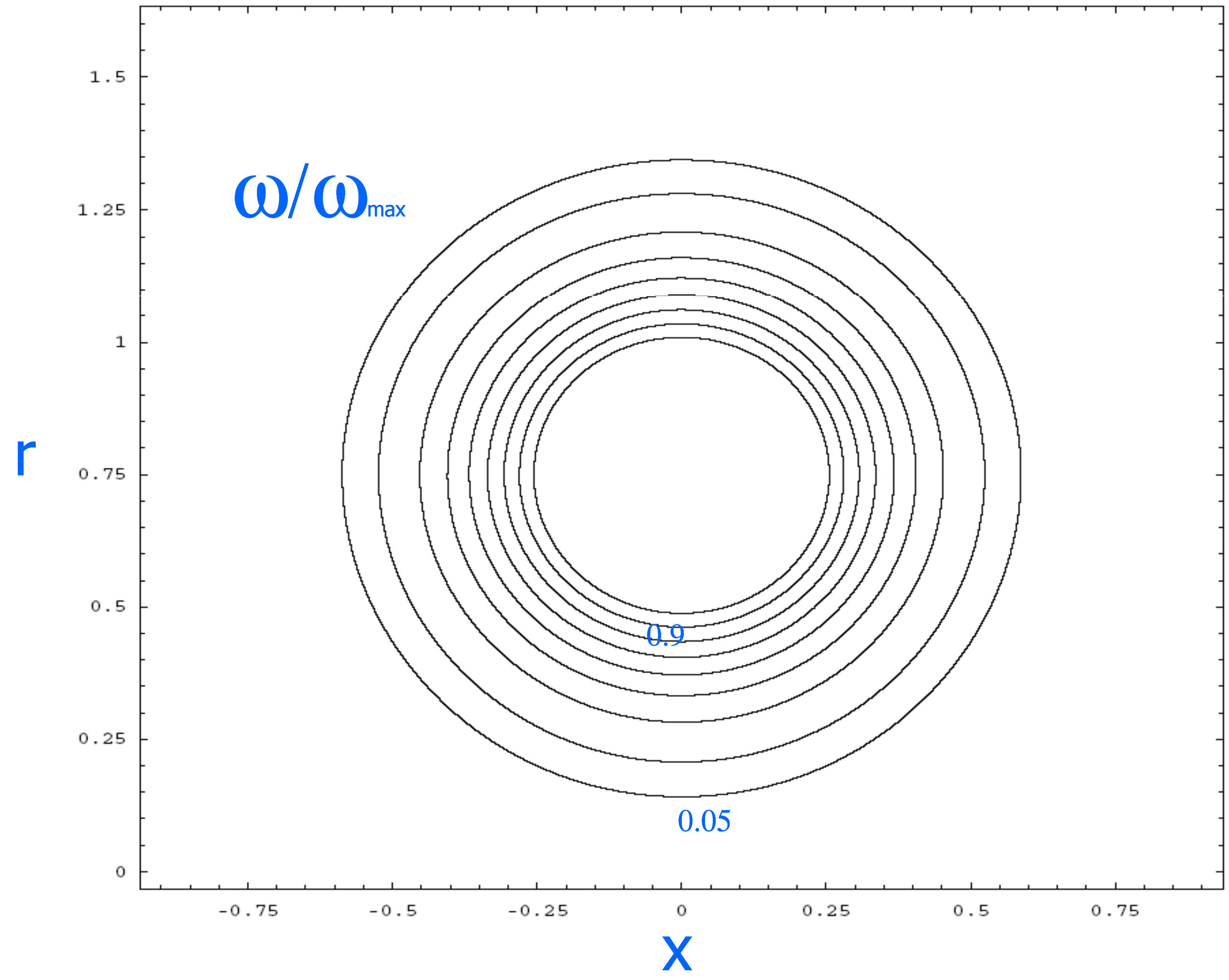




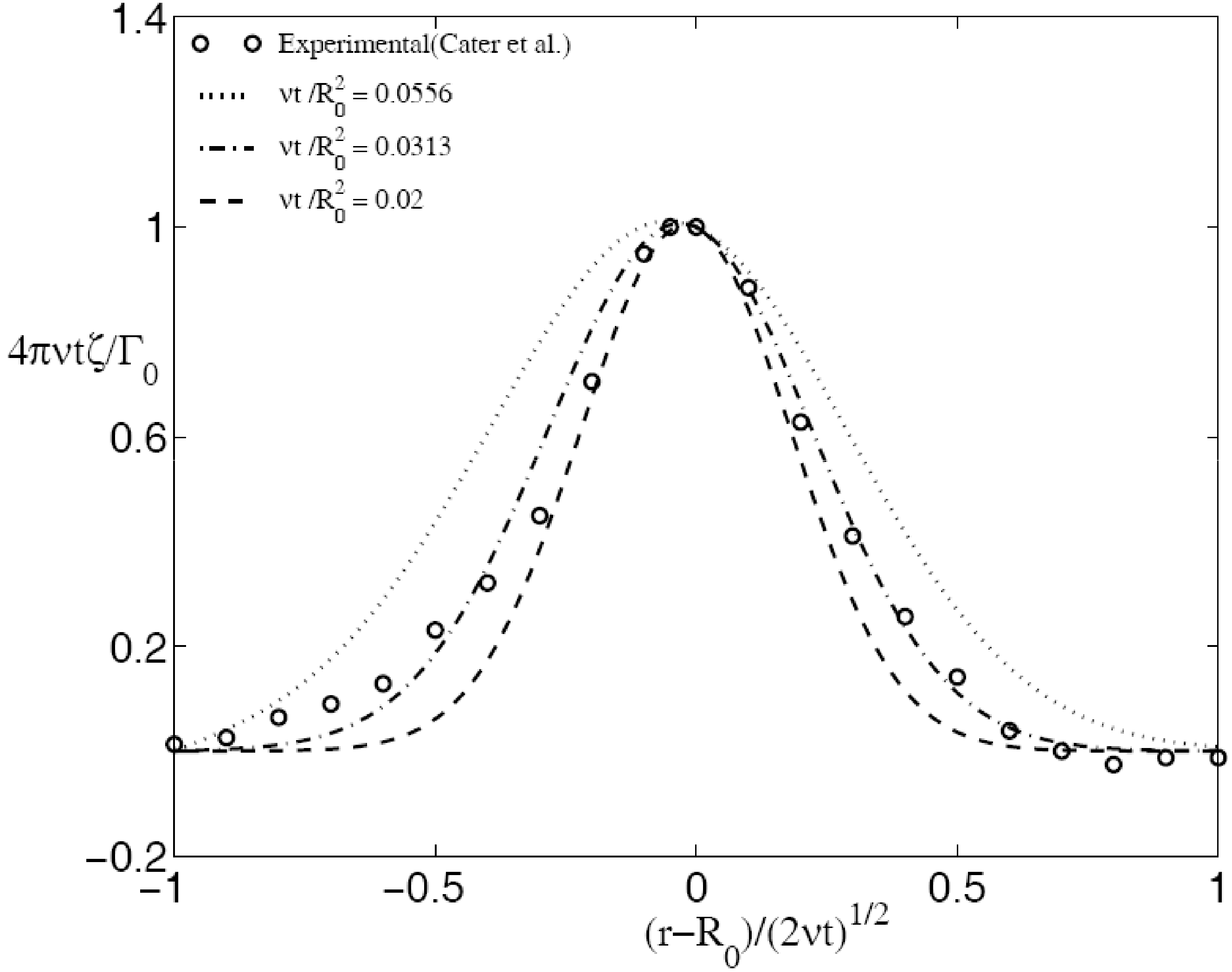
Testing of the predictions for the vorticity distribution and stramfuction

Isolines of vorticity distribution

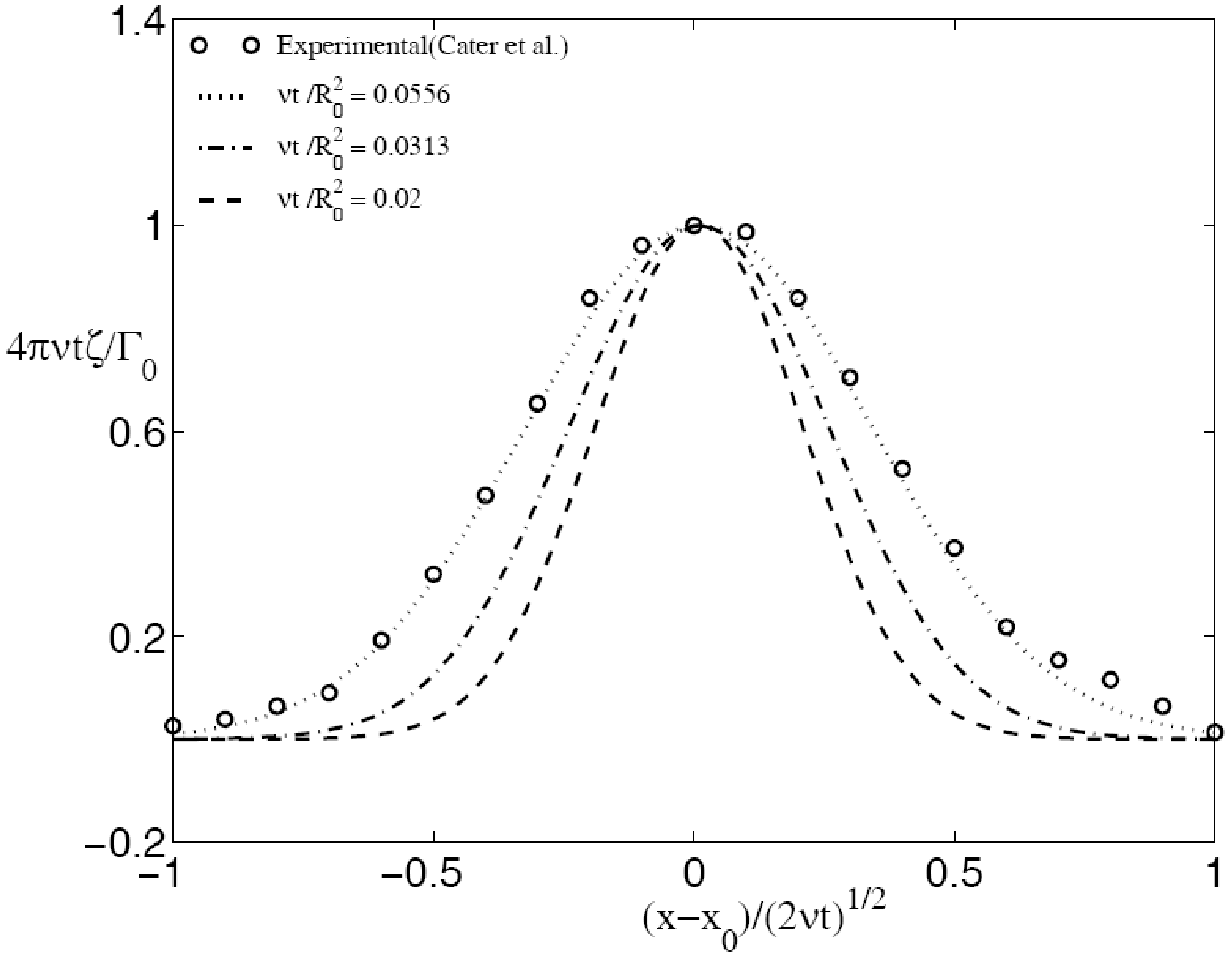
$$\omega = exp\left(-\frac{1}{2}\left(r^2 + x^2 + R^2\right)\frac{\tau^2}{R^2}\right)I_1\left(r\frac{\tau^2}{R}\right),$$



Comparison of radial vorticity distribution with experimental data by Cater et.al.(JFM,2004)



Comparison of axial vorticity distribution with experimental data by Cater et.al.(JFM,2004)



Streamfunction

$$(f = \Phi / \sigma) \leftarrow \overline{f}$$

$$\Psi = \frac{2\varsigma_0 \ell^3 \sigma}{(2\pi)^{1/2}} \int_0^\infty \int_0^\infty \frac{\mu \exp\left(\frac{-\mu^2 - \alpha^2}{2}\right)}{\mu^2 + \alpha^2} \mathrm{J}_1(\tau\mu) \mathrm{J}_1(\sigma\mu) \cos(\alpha\eta)\, d\mu\, d\alpha$$

$$\Psi = \frac{M \sigma}{4\pi R_0} \int_0^\infty \mathrm{F}(\mu, \eta) \mathrm{J}_1(\tau\mu) \mathrm{J}_1(\sigma\mu)\, d\mu$$

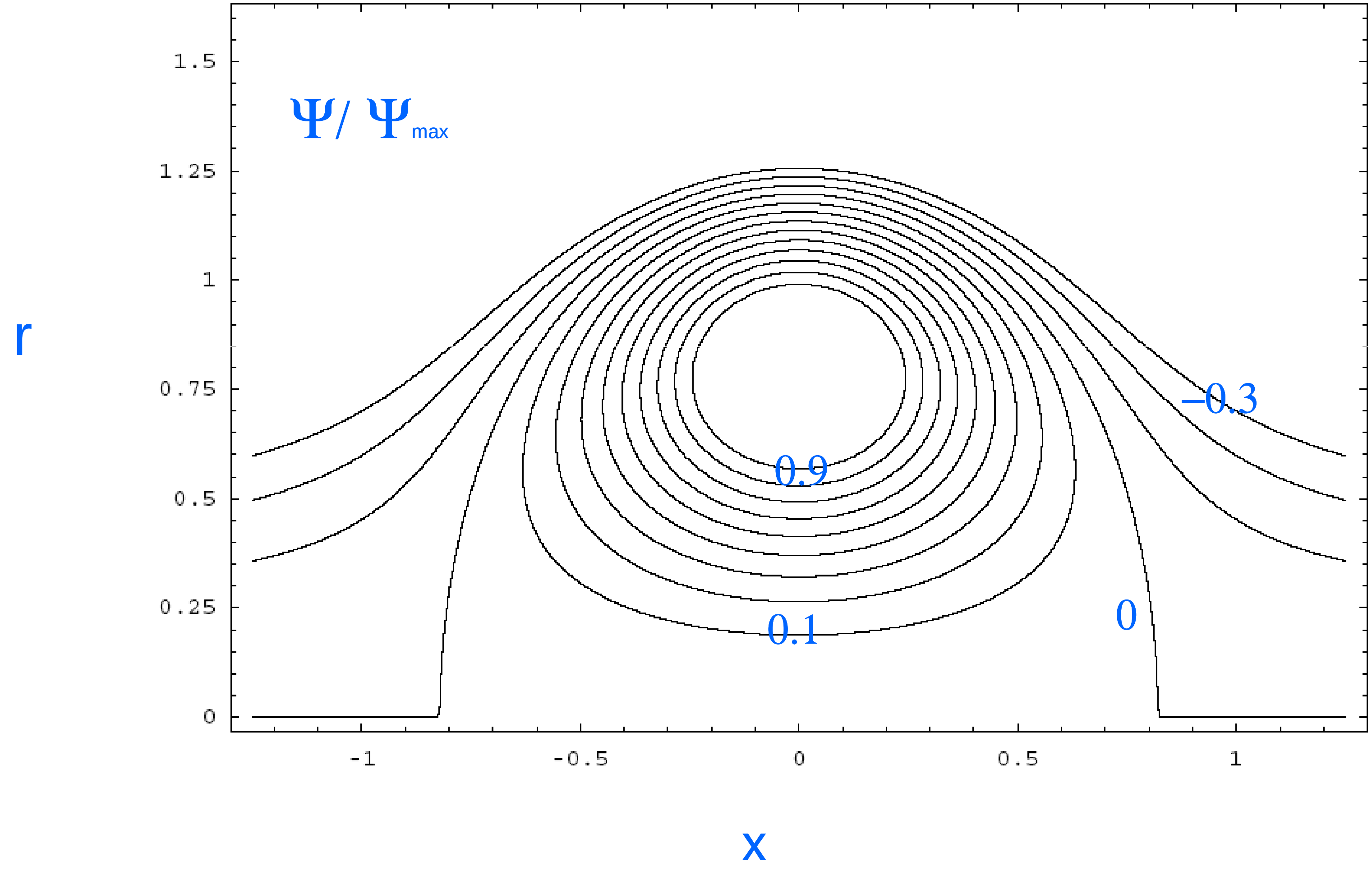
$$\mathrm{F}(\mu,\eta) = \mathrm{G}(\mu,\eta) + \mathrm{G}(\mu,-\eta), \mathrm{G}(\mu,\eta) = \exp(\eta\mu)\left(1 - \mathrm{erf}\left(\frac{\mu+\eta}{\sqrt{2}}\right)\right)$$

$$u = \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma} = \frac{\sqrt{\pi}}{2\sqrt{2}} \int_0^\infty \mu^2 \mathrm{F}(\mu, \eta) \mathrm{J}_1(\tau\mu) \mathrm{J}_1(\sigma\mu)\, d\mu$$

$$v = -\frac{1}{\sigma}\frac{\partial \Phi}{\partial \eta} = -\frac{\sqrt{\pi}}{2\sqrt{2}} \int_0^\infty \mu\{-\mathrm{G}(\mu,-\eta) + \mathrm{G}(\mu,\eta)\} \mathrm{J}_1(\tau\mu) \mathrm{J}_1(\sigma\mu)\, d\mu$$

Using relations

$$\Psi = \Psi_0 - U_{tr} r^2 / 2 \qquad \sigma = \frac{r}{\ell} = \frac{r\tau}{R}, \eta = \frac{x}{\ell} = \frac{x\tau}{R}$$



θ

Two models of vortex rings

Norbury- Fraenkel model: impulse, circulation, energy, translational velocity. Rings classified by a nondimensional mean core radius α

Our model: constant impulse, circulation, energy, translational velocity. Rings classified by parameter $\tau = Ro/\ell$, ℓ – core radius

Comparison of the models

Recently Danaila&Helie have presented numerical simulation of the postformation evolution of a laminar vortex ring (Phys. Fluids, 20, 073602, 2008). These authors have shown that the presented model offers, in addition to a good prediction of integral quantities, a more accurate description of the vortex ring topology in comparison with the Norbury-Fraenkel model.

Signatures of the vortex ring for different models from paper by Danaila&Helie

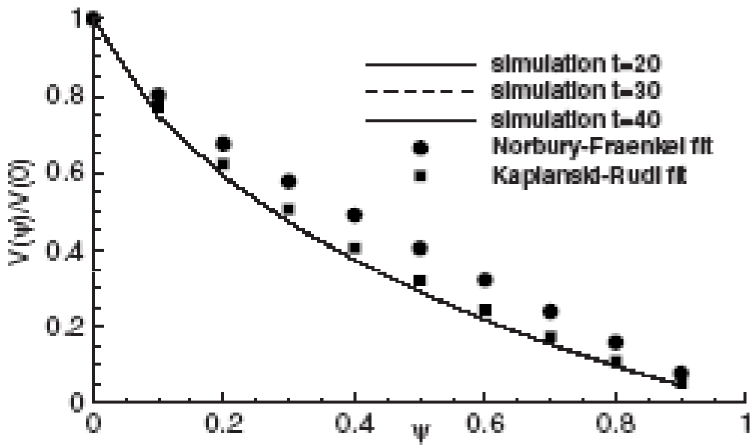


FIG. 13. Fit to ideal vortex models. Signature (Ref. 34) of the vortex ring normalized by its maximum value.

APPLICATION OF THE VORTEX RING MODEL: FORMATION NUMBER

Schematic view of vortex ring generator (Gharib *et al.*,1998)

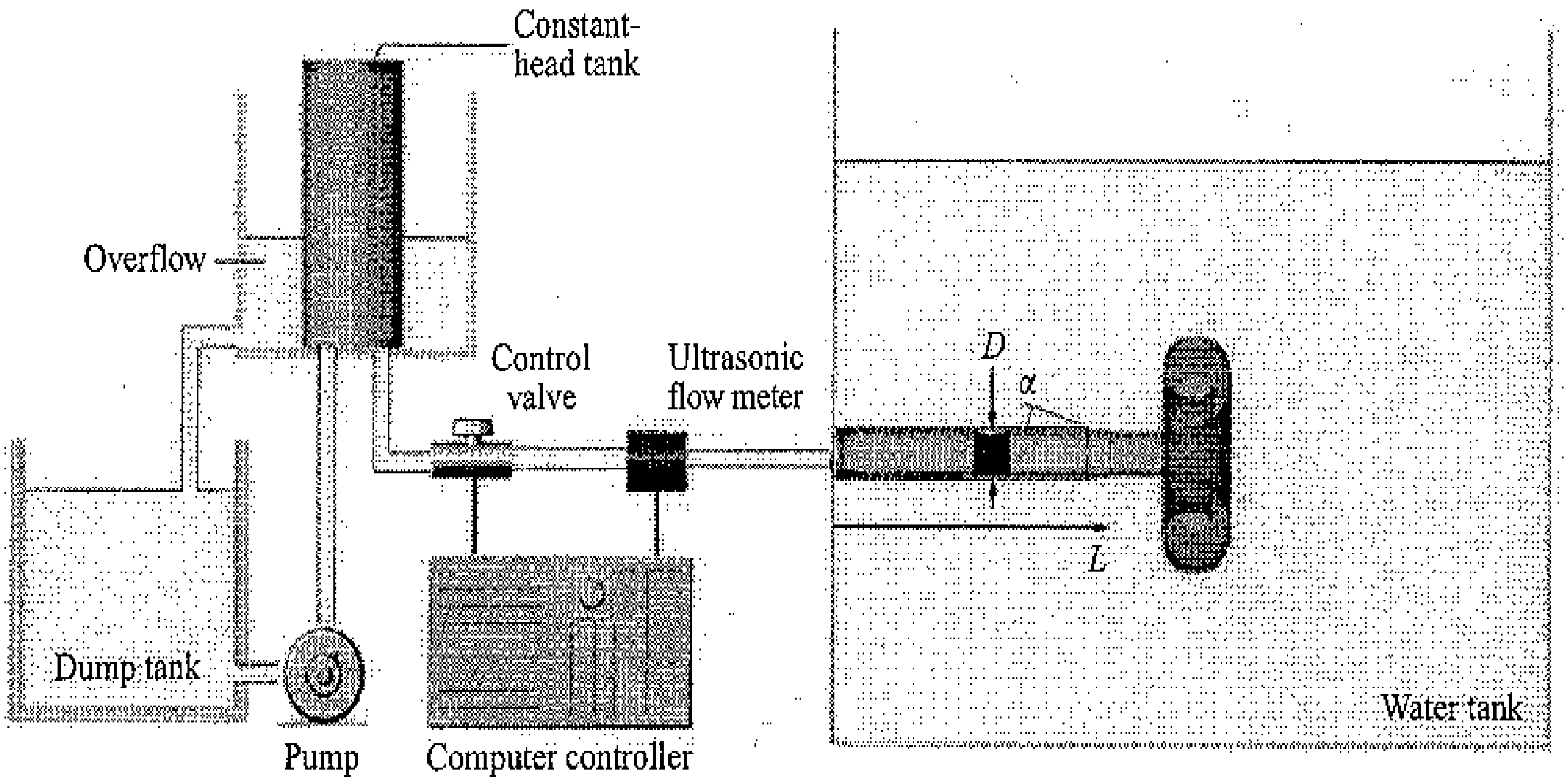


FIGURE 1. General schematic of vortex ring generator.

Formation stage (Gharib *et al.*,1998)

L/D<4

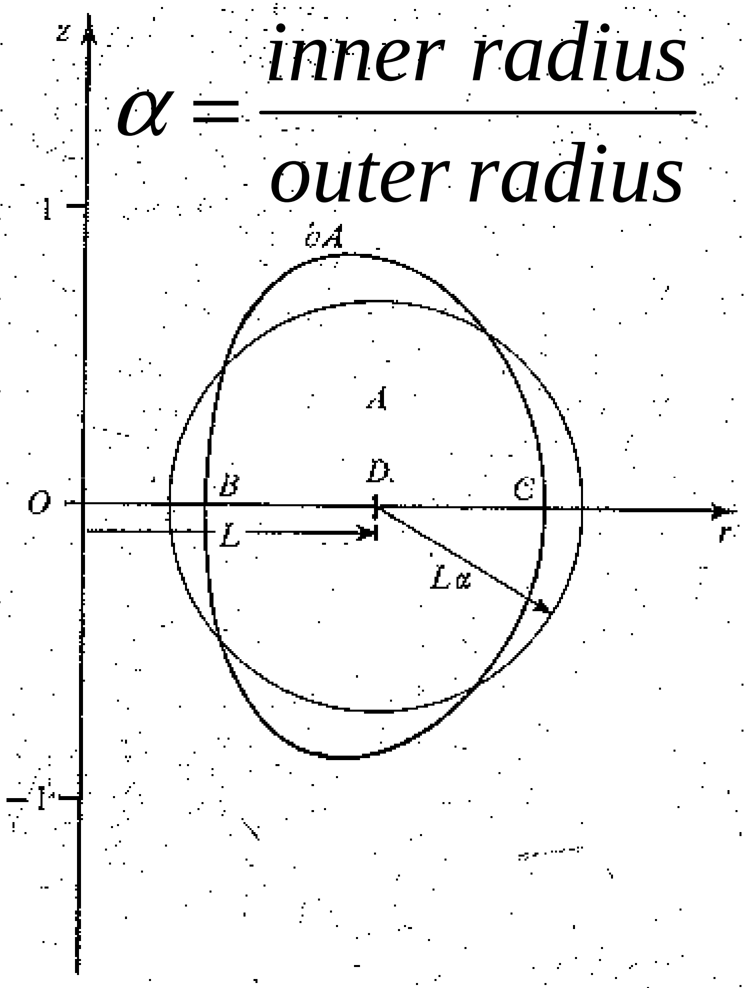
L/D≈4

L/D>4

FIGURE 3. Visualization of vortex rings at $X/D \approx 9$ for (a) $L_m/D = 2$, $Re \approx \Gamma/\nu \approx 2800$; (b) $L_m/D = 3.8$, $Re \approx 6000$; and (c) $L_m/D = 14.5$. Picture is taken at $\bar{U}_p t/D = L/D = 8$. All three cases were generated by an impulsive piston velocity depicted in figure 2.

Norbury family of steady vortex rings with uniform vorticity distributions (1973)

$$\alpha = \frac{inner\ radius}{outer\ radius} \rightarrow 1/\tau$$



Vorticity distribution for a ring at stage of formation from experimental data of Fabris&Liepman, 1997 (Re=3700).

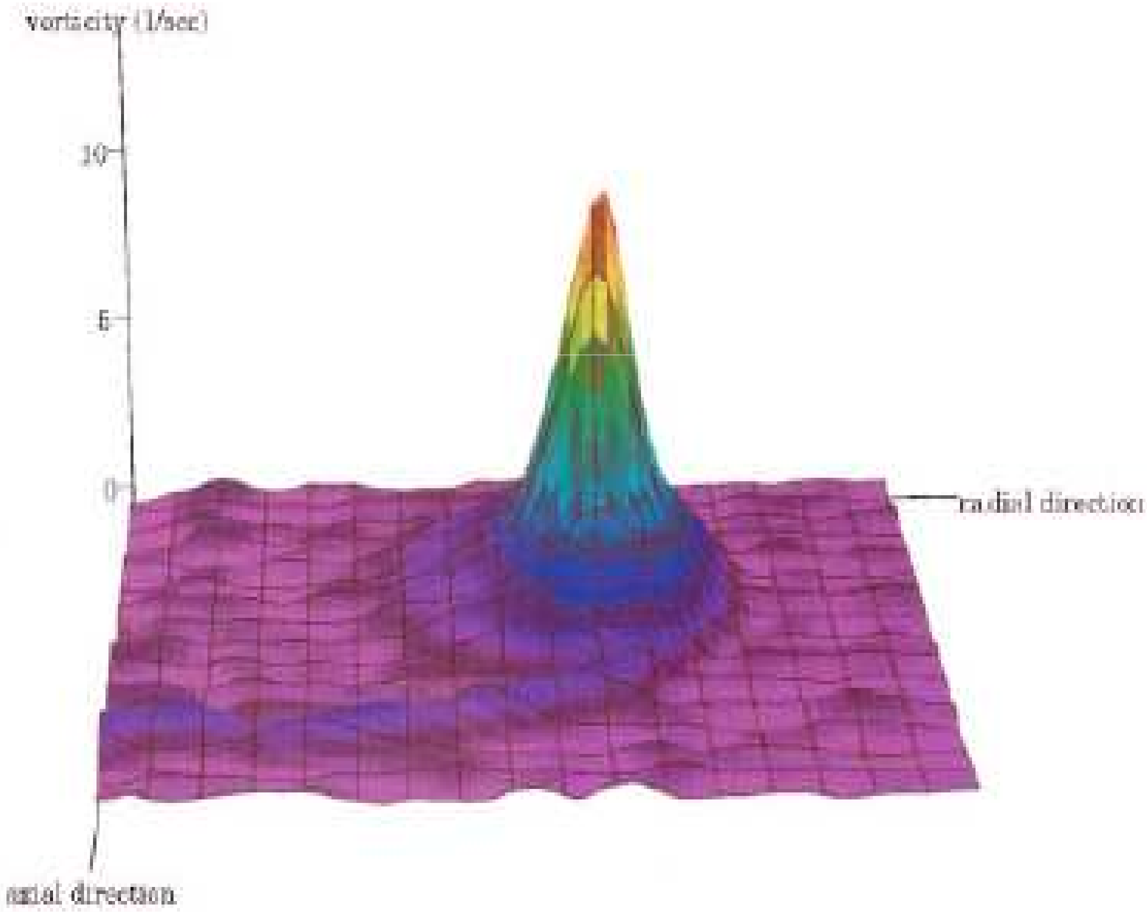
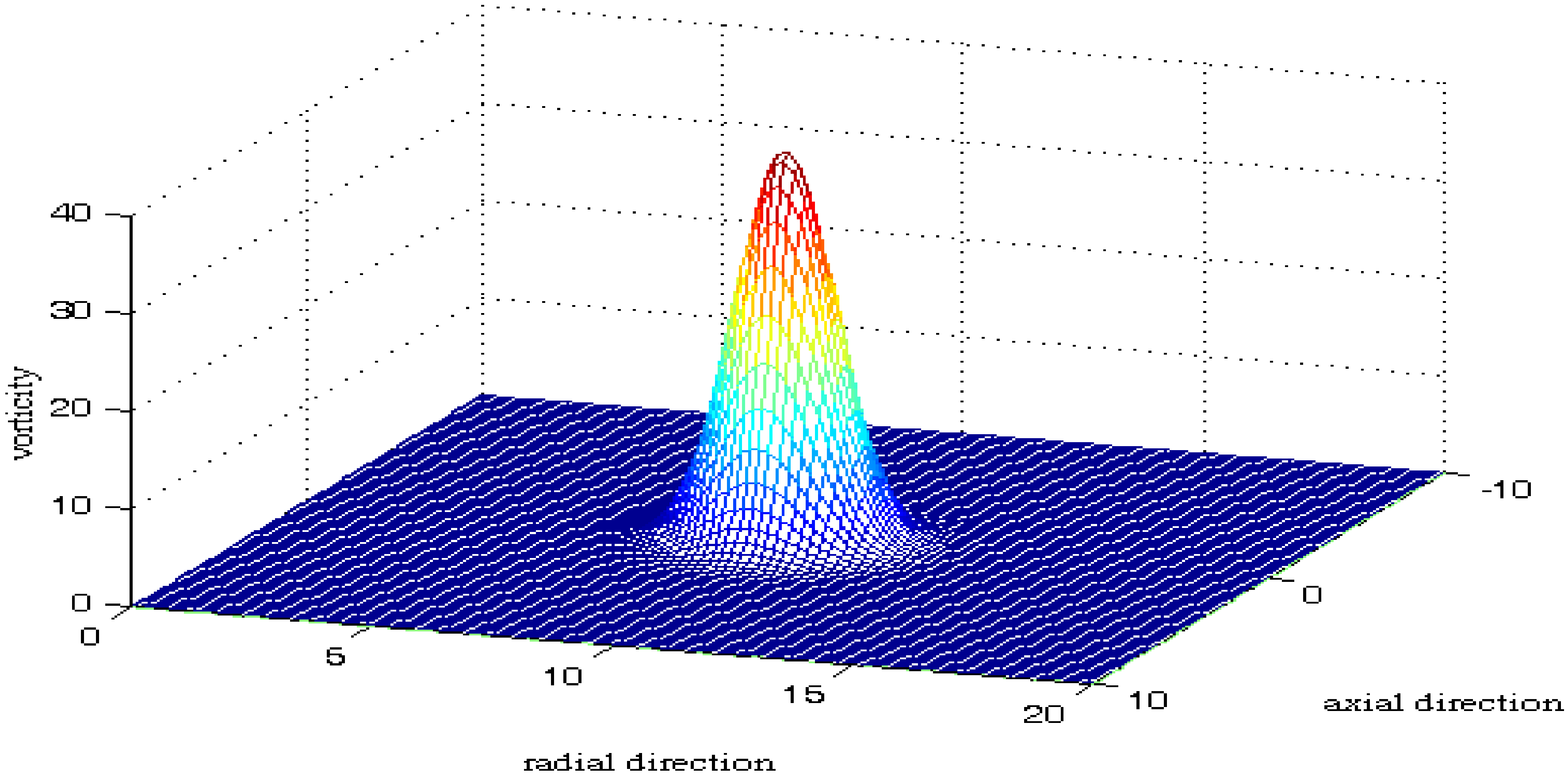
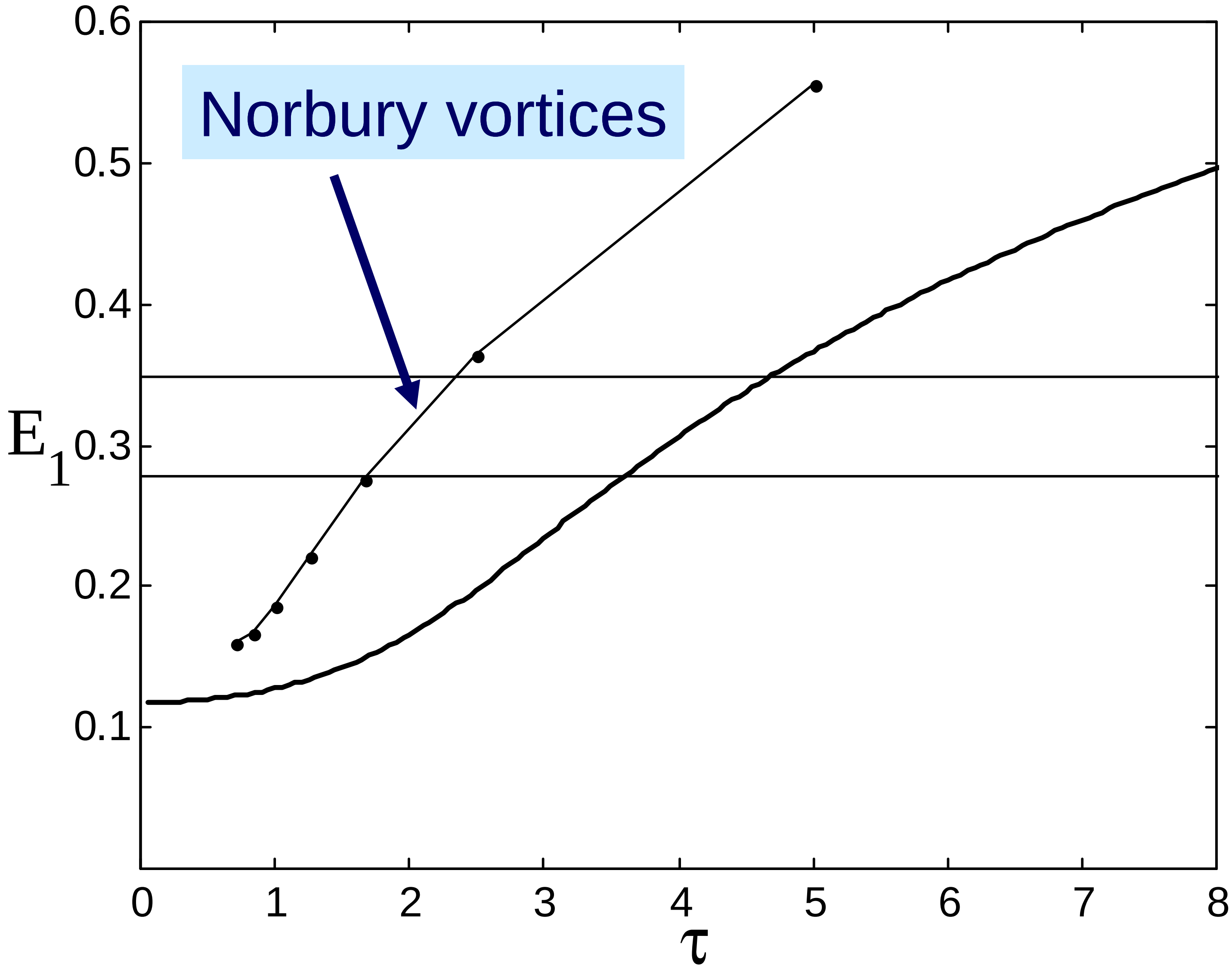


FIG. 4. Vorticity distribution for a ring of Reynold's number 3700 (same stroke, piston velocity 2.5 cm/s, and impulsive acceleration, measurement points are spaced ever 2 mm). Only the right core is shown. The ring propagates forward and the centerline is at the left.

Vorticity distribution for t=10 (our model)

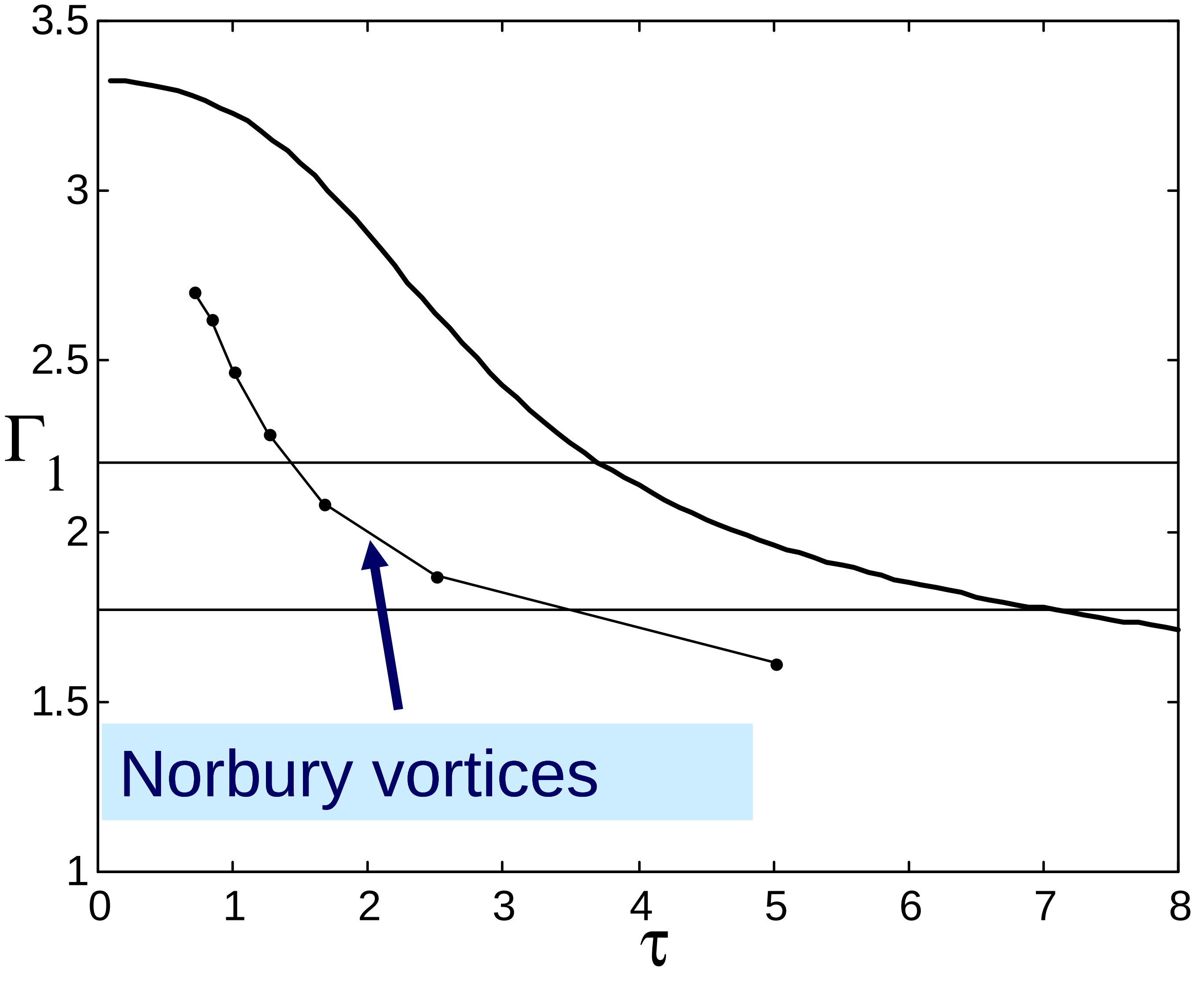


Dimensionless energy $E_1 = \dfrac{E_m}{\Gamma_m^{3/2} I_m^{1/2}}$ us τ

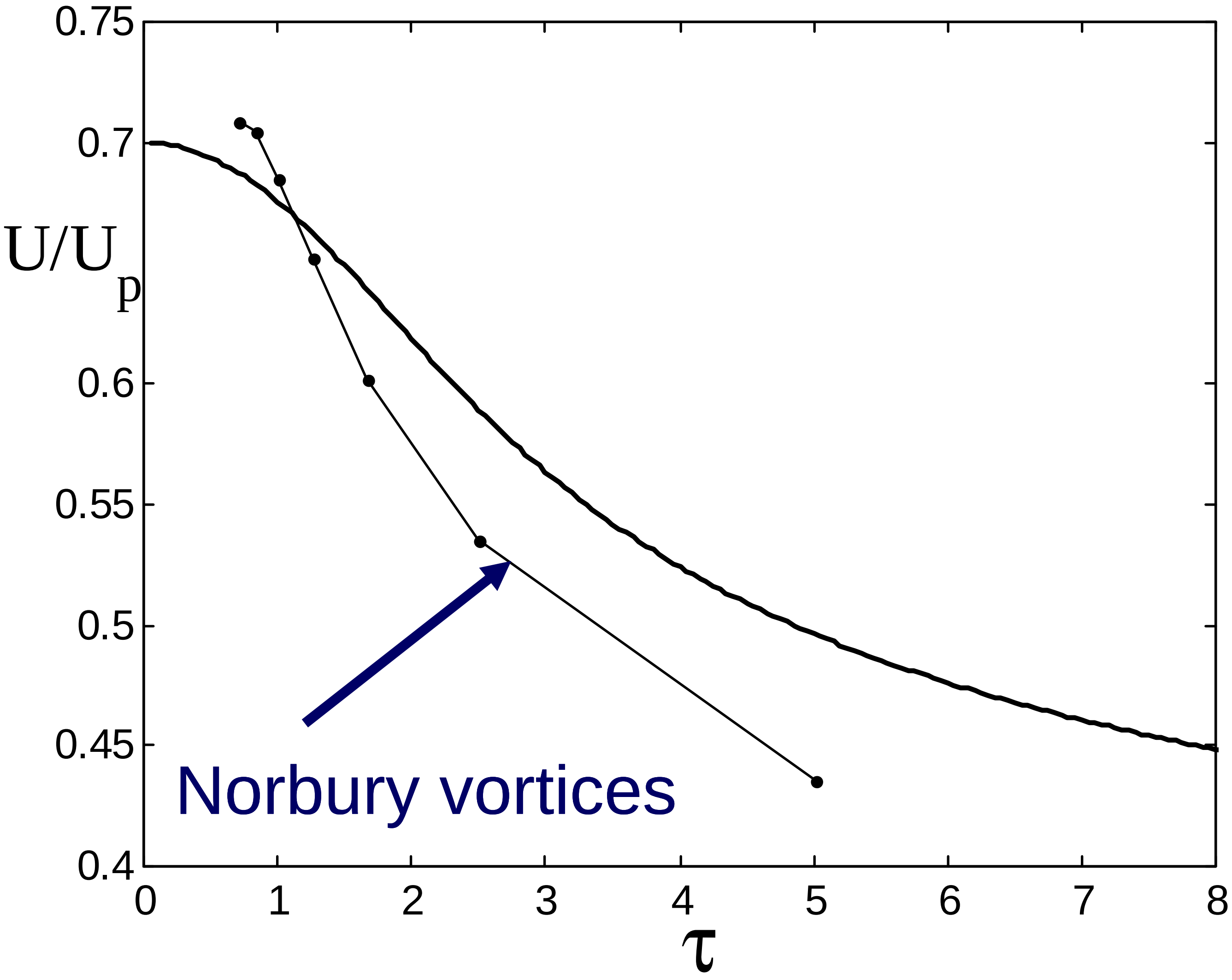


Dimensionless circulation versus τ

$$\Gamma_1 = \frac{\Gamma_m}{I_m^{1/3} U_m^{2/3}}$$



Ratio of the translational velocity to the ejection velocity $\frac{U}{U_p} = \frac{U_m I_m}{2 E_m}$ versus τ



Slug model
(earlier used by Mohseni&Gharib, 1998 and by Linden&Turner, 2001)

$$\Gamma = \frac{1}{2} L U_p ,$$

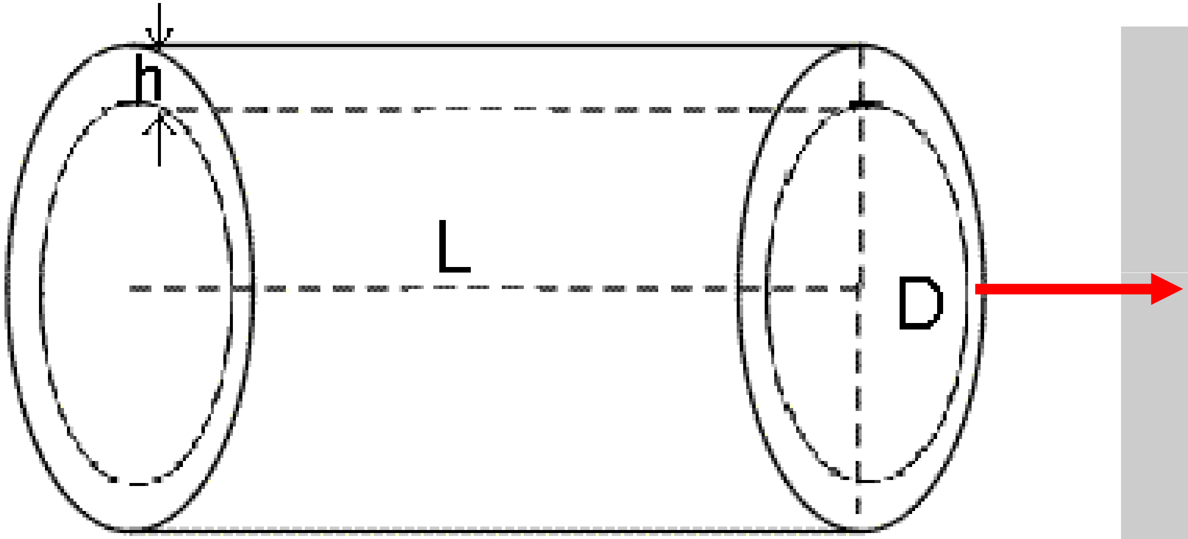
$$I = \frac{1}{4}\pi D^2 L U_p = \frac{1}{2}\pi D^2 \Gamma$$

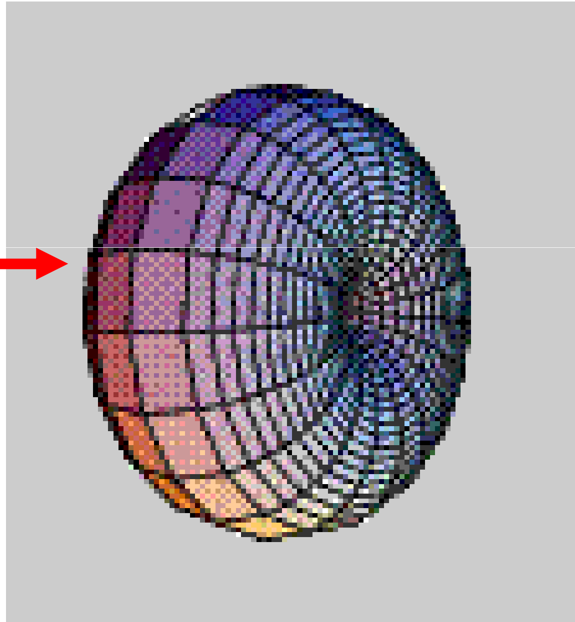
$$\frac{L}{D} = \sqrt{\frac{\pi}{2}} \frac{\Gamma^{3/2} I^{1/2}}{E} = \sqrt{\frac{\pi}{2}} \frac{1}{E_1},$$

$$E = \frac{1}{8}\pi D^2 L U_p^2 = \frac{U_p}{2} I$$

$$E_1 = \frac{E_m}{\Gamma_m^{3/2} I_m^{1/2}}$$

Lower estimate for L/D
(geometric approach, based on the equalization of the volumes for the cylindrical vortex sheet and toroidal vortex ring)





$$L/D \geq 2\pi(\ell^2 R_0/D^3)\frac{1}{(z-z^2)}, z = h/D$$

if $h = R_0 = \ell = D/2$ then $L/D \geq \pi$

PREDICTION OF THE 'FORMATION' NUMBER (kinematic approach, Shusser&Gharib, 2000)

Equation for τ at the pinch–off, based on the slug model

$$\alpha(\tau) = \frac{B(\tau)}{2N(\tau)\sqrt{\pi}},$$

where

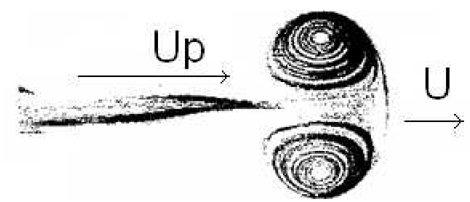
$$\alpha = \frac{E}{\rho I \Gamma^3},$$

$$B = U\sqrt{\frac{\pi I}{\rho \Gamma^3}},$$

$$b = R_0\sqrt{\frac{\rho \pi \Gamma}{2I}},$$

$$N = \frac{U}{U_p} = \frac{UI}{2E}.$$

Criterion for the pinch-off



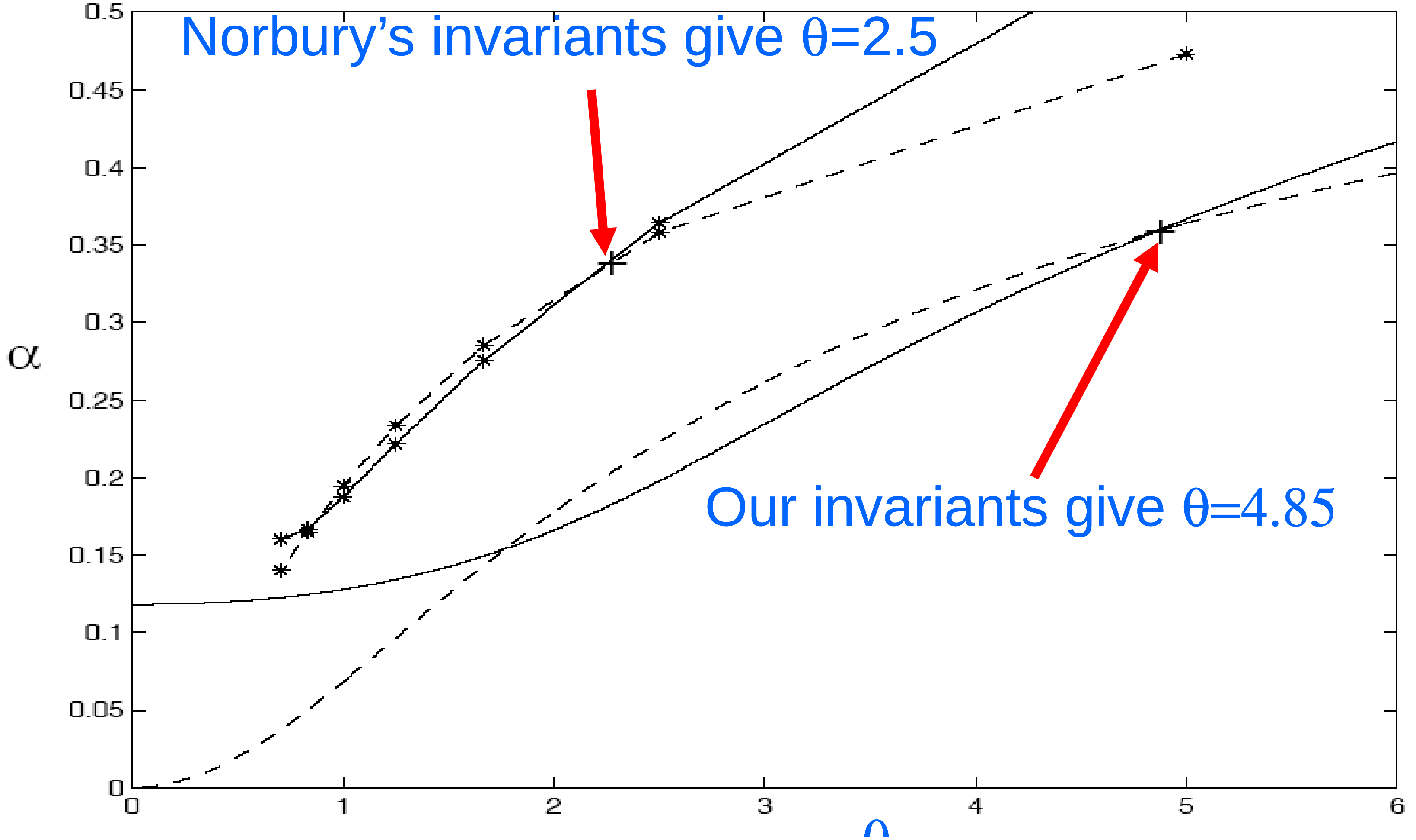
$$U = \frac{D^2}{4R_0^{\ 2}} U_p$$

FORMATION NUMBER

$$\frac{L}{D} = \frac{\pi\sqrt{2}}{4b^2B} = 3.5$$

B=0.6907,b =0.6775, Norbury's data

B=0.6350,b =0.7071, our data



SUMMARY

Thank you for your attention